Math 3124A/9024A Assignment 1

University of Western Ontario

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1. (Bak–Newman E1.1) Express in the form $a + bi$:

   a. $\dfrac{1}{6+2i}$

   b. $\dfrac{(2+i)(3+2i)}{1-i}$

   c. $\left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)^4$

   d. $i^2, i^3, i^4, i^5, \ldots$

2. (Bak–Newman E1.4) Prove the following identities:

   a. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.

   b. $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

   c. $\overline{P(z)} = P(\bar{z})$, for any polynomial with *real* coefficients.

   d. $\bar{\bar{z}} = z$.

3. (Bak–Newman E1.5) Suppose P is a polynomial with real coefficients. Show that $P(z) = 0$ if and only if $P(\bar{z}) = 0$ (i.e., zeroes of "real" polynomials come in conjugate pairs).

4. Using the definitions from the text, show that $\lim_{k\to\infty} z_k = 0$ whenever the series $\sum_{k=1}^{\infty} z_k$ is convergent.

5. Using the definitions from the text, show that a set S is closed in $\mathbb{C}$ iff $\{z_n\}_{n=1}^{\infty} \subseteq S$ and $\lim_{n\to\infty} z_n = z$ imply $z \in S$; that is, show

$$S \text{ is closed} \quad \Longleftrightarrow \quad \forall \{z_n\}_{n=1}^{\infty} \subseteq S \text{ with } \lim_{n\to\infty} z_n = z \in \mathbb{C} \implies z \in S$$

6. (Bak–Newman E1.22) Prove that a polygonally connected set is connected.

7. (Bak–Newman E1.26) Let P be a nonconstant polynomial in z. show that $P(z) \to \infty$ as $z \to \infty$.

8. [**MATH 9024 STUDENTS ONLY**] Let S be a set. An *order* on S is a relation, denoted by "$\prec$", with the following two properties:

   (i) If $a \in S$ and $b \in S$ then one and only one of the statements

   $$a \prec b, \quad a = b, \quad b \prec a$$

   is true.

   (ii) If $a, b, c \in S$, if $a \prec b$ and $b \prec c$, then $a \prec c$.

   The statement "$a \prec b$" may be read as "a is less than b" or "a is smaller than b" or "a precedes b". It is sometimes convenient to write "$b \succ a$" when meaning "$a \prec b$". It is a natural generalization of the standard ordering "$<$" on $\mathbb{R}$.

   An *ordered field* is a *field* F which is also an *ordered set*, such that

   (i) $a + b \prec a + c$ if $a, b, c \in F$ and $b \prec c$,

   (ii) $ab \succ 0$ if $a \in F$, $b \in F$, $a \succ 0$, and $b \succ 0$.

   (iii) If $a \succ 0$ and $b \prec c$, then $ab \prec ac$.

   (iv) If $a \prec 0$ and $b \prec c$, then $ab \succ ac$.

   It is not *too* difficult to see that (iii) and (iv) are consequences of the other conditions. If $a \succ 0$, we call a *positive*; if $a \prec 0$, a is *negative*.

   Exercises:

   a. Write $z, w \in \mathbb{C}$ as $z = a + bi$ and $w = c + di$. Define $z \prec w$ if $a < c$, and also if $a = c$ but $b < d$. Prove this turns $\mathbb{C}$, as a set, into an ordered set. (This type of order relation is called *dictionary order*, or *lexicographic order*, for obvious reasons.)

   b. Prove that no order can be defined in $\mathbb{C}$ that turns it into an ordered field. *Hint:* -1 is a square.