Math 3124A/9024A Assignment 2

University of Western Ontario

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1. (Bak–Newman E2.3) By comparing coefficients or by use of the Cauchy–Riemann equations, determine which of the following polynomials are analytic.

   a. $P(x+iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y)$.

   b. $P(x+iy) = x^2 + iy^2$.

   c. $P(x+iy) = 2xy + i(y^2 - x^2)$.

2. (Bak–Newman E2.4) Show that no nonconstant analytic polynomial can take imaginary values only.

3. (Bak–Newman E2.5) Find the derivative $P'(z)$ of the polynomials which are analytic in Question 1 (Bak–Newman E2.3). You may use any proposition/theorem in Chapters 1 or 2 of the text to do so. Show in each case that $P'(z) = P_x$

4. (Bak–Newman E2.6) Prove Proposition 2.5 in the text. [*Hint*: You don't have to prove the quotient rule form scratch; just prove the product rule and use it in the proof of the quotient rule. Note also that the the product rule, once proved, implies

$$0 = (1)' = \left(g \cdot \frac{1}{g}\right)'(z) = \frac{g'(z)}{g(z)} + g(z)\left(\frac{1}{g}\right)'(z),$$

   from which one easily gets a formula for $(1/g)'(z)$ in terms of g.]

5. (Bak–Newman E2.7) Prove Proposition 2.6 in the text. [*Hint*: Prove it for monomials and then apply Proposition 2.5].

6. (Bak–Newman E2.10) Suppose $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence R. Using any properties of the limit supremum that you learned in Math 2122A/B (or equivalent), find the radius of convergence of

   a. $\sum_{n=0}^{\infty} n^p c_n z^n$,

   b. $\sum_{n=0}^{\infty} |c_n| z^n$,

   c. $\sum_{n=0}^{\infty} c_n^2 z^n$.

7. (Root test is stronger than ratio test)

   a. Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers and

   $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \tag{1}$$

   Show then that $\lim_{n\to\infty} a_n^{1/n} = L$. This means that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ can be determined through calculation of the limit above, if it exists. [*Hint*: Convergence of (1) implies that for every $\varepsilon > 0$ there exists a N so that we have both

   $$\frac{a_{n+1}}{a_n} < L + \varepsilon \quad \text{and} \quad L - \varepsilon < \frac{a_{n+1}}{a_n}$$

   whenever $n \geq N$. Furthermore, if $n > N$, then

   $$\frac{a_n}{a_N} = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+2}}{a_{N+1}} \frac{a_{N+1}}{a_N}.$$

   Use this to show $\lim_{n\to\infty} a_n^{1/n} = L$.]

   b. On the other hand, Consider

   $$a_n = \begin{cases} 2^{-n}, & \text{if } n \text{ is odd} \\ 2^{-n+2}, & \text{if } n \text{ is even.} \end{cases}$$

   Show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist, but $\limsup_{n\to\infty} a_n^{1/n}$ does. Therefore the root test is insufficient in general for determining the radius of convergence of a power series.

8. (Bak–Newman E2.21) Show that there is no power series $f(z) = \sum_{n=0}^{\infty} C_n z^n$ converging near the origin such that

   i. $f(z) = 1$ for $z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

   ii. $f'(0) > 0$.

9. [**MATH 9024 STUDENTS ONLY**]. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a power series that converges on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Show that if $\sum_{n=0}^{\infty} |f^{(n)}(0)| < \infty$, then f is actually convergent on the entire complex plane; that is, it has radius of convergence $R = \infty$.

10. [**MATH 9024 STUDENTS ONLY**]. Based on class discussions, the collection of functions f which can be represented as a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ convergent in the unit disk forms a commutative ring under the operation of addition and multiplication. Prove that this ring is an integral domain. In other words, prove that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is not the zero function, and if $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is not the zero function, then the product fg is not the zero function.