Math 3124A/9024A Assignment 3

University of Western Ontario

Fall 2023

1. (Bak–Newman E3.3) Prove that the composition of differentiable functions is differentiable. That is, if f is differentiable at z, and if g is differentiable at $f(z)$, then $g \circ f$ is differentiable at z. [Hint: A first attempt may be to write

$$\begin{aligned}(g \circ f)'(z) &= \lim_{w \to z} \frac{g(f(z)) - g(f(w))}{z - w} \\ &= \lim_{w \to z} \left[\frac{g(f(z)) - g(f(w))}{f(z) - f(w)} \frac{f(z) - f(w)}{z - w} \right].\end{aligned}$$

and since f is continuous at z because it is differentiable there, we have

$$(g \circ f)'(z) = g'(f(z)) f'(z).$$

There is however one problem with this approach: *What if f is zero near z?* You can either treat the case where f is zero near z separately (make sure you negate the statement "f is zero near z" correctly!), or use the following, different approach. Set $w = f(z)$ and define

$$\begin{aligned}\varepsilon(\eta) &:= \frac{f(z) - f(\eta)}{z - \eta} - f'(z) \\ \delta(\zeta) &:= \frac{g(w) - g(\zeta)}{w - \zeta} - g'(w)\end{aligned}$$

so that $\lim_{\eta \to z} \varepsilon(\eta) = 0$ and $\lim_{\zeta \to w} \delta(\zeta) = 0$. Then, starting with $g(f(z)) - g(f(\eta))$, use the definitions of ε and δ above.]

2. (Bak–Newman E3.8) Find all analytic functions $f = u + iv$ with $u(x, y) = x^2 - y^2$.

3. (Bak–Newman E3.11) Define e^z by

$$e^z = e^x \cos y + i e^x \sin y.$$

   a. Show that e^z is entire by verifying the Cauchy–Riemann equations for its real and imaginary parts.

b. Prove that

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

4. (Bak–Newman E3.15) Verify the identities

   a. $\sin 2z = 2\sin z \cos z$;
   b. $\sin^2 z + \cos^2 z = 1$;
   c. $(\sin z)' = \cos z$,

   where $\sin z$ and $\cos z$ are defined by

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$
$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).$$

5. [**MATH 9024 STUDENTS ONLY**] (Bak–Newman E3.21) Show that the power series

$$f(z) = 1 + z + \frac{z^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is equal to e^z, as defined above. [*Hint*: First show that $f(z)f(w) = f(z+w)$, then show

$$\begin{aligned} f(z) &= e^x \\ f(iy) &= \cos y + i \sin y \end{aligned}$$

using the power series representations for e^x, $\cos x$, and $\sin x$ as functions from $\mathbb{R}$ to $\mathbb{R}$.]