Math 3124A/9024A Assignment 5

University of Western Ontario

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1. We have used a few times the following fact: Suppose that f is a continuous complex-valued function on a compact (closed and bounded) set K in $\mathbb{C}$, then $|f|$ attains a maximum on K, i.e., there exists a $z \in K$ such that $|f(w)| \leq |f(z)| < \infty$ for all $w \in K$. Prove this fact using the following outline:

   (i) Set $M := \sup_{z \in K} |f(z)|$ and use the definition of the supremum to show that there exists a sequence $\{f(z_n)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f(z_n) = \alpha$ with $|\alpha| = M$. Keep in mind that we do not know M is finite at this point in the proof.

   (ii) Note that the sequence $\{z_n\}_{n=1}^{\infty} \subseteq K$ does not necessarily converge. Apply the Bolzano–Weierstrass theorem to show that $\{z_n\}_{n=1}^{\infty} \subset K$ admits a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ converging to a point $z \in \mathbb{C}$. Note that the Bolzano–Weierstrass theorem only applies to sequences in $\mathbb{R}$, so be careful to justify its usage.

   (iii) Justify that $z \in K$ and $f(z) = \alpha$, completing the proof.

2. (Bak–Newman E5.5) (A Generalization of the Cauchy Integral Formula). Show that

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_C \frac{f(w)}{(w-a)^{k+1}} \mathrm{d}w \quad \text{for } k = 1, 2, \ldots,$$

   where C surrounds the point a and f is entire.

3. (Bak–Newman E5.6)

   (a) Suppose an entire function f has $|f| \leq M$ along $|z| = R$. Show that the coefficients C_k in its power series expansion about 0 satisfy

$$|C_k| \leq \frac{M}{R^k}.$$

   (b) Suppose a polynomial is bounded by 1 in the unit disk. Show that all its coefficients are bounded by 1.

4. (Bak–Newman E5.7) (An alternate proof of the generalized Liouville's Theorem). Suppose that $|f(z)| \leq A + B|z|^k$ and that f is entire. Show that then all the coefficients $C_j, j > k$ in its power series expansion are 0. See the previous exercise.

5. (Bak–Newman E5.8) Suppose f is entire and $|f(z)| \leq A + B|z|^{3/2}$. Show that f is a linear polynomial.

6. (Bak–Newman E5.10) Prove that a nonconstant function cannot satisfy the two equations

   i. $f(z+1) = f(z)$

   ii. $f(z+i) = f(z)$.

   for all $z \in \mathbb{C}$. [*Hint*: Show that a function satisfying both equalities would be bounded.]

7. (Bak–Newman E5.14) Show that α is a zero of multiplicity k if and only if
$$P(\alpha) = P'(\alpha) = \ldots = P^{(k-1)}(\alpha) = 0 \qquad \text{and} \qquad P^{(k)}(\alpha) \neq 0$$

8. [**MATH 9024 STUDENTS ONLY**] Determine every entire function f having the property that $|f(z)| \leq |\sin(z)|$ for all values $z \in \mathbb{C}$. Justify every step carefully.