Math 3124A/9024A Assignment 6

University of Western Ontario

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1. (Bak–Newman E.6.1) Find a power series expansion for $1/z$ around $z = 1 + i$.

2. (Bak–Newman E.6.2) Find a power series, centered at the origin, for the function $f(z) = \dfrac{1}{1 - z - 2z^2}$ by first using partial fractions to express $f(z)$ as a sum of two simple rational functions.

3. (Bak–Newman E.6.4) Show that if f is analytic in $|z| \leq 1$, there must be some positive integer n such that $f(1/n) \neq \frac{1}{n+1}$. [*Hint:* Consider using uniqueness theorem].

4. (Bak–Newman E.6.5) Prove that $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.

5. (Bak–Newman E.10) Find the maximum and minimum moduli of $z^2 - z$ in the disk $\{|z| \leq 1\}$.

6. (Bak–Newman E.6.11) (A proof, due to Landau, of the maximum modulus theorem) Suppose f is analytic inside and on a circle C with $|f(z)| \leq M$ on C, and suppose z_0 is a point inside C. Use Cauchy's integral formula to show that $|f(z_0)|^n \leq KM^n$, where K is independent of n, and deduce that $|f(z_0)| \leq M$.

7. [**MATH 9024 ONLY**]. Suppose f is an entire function and that $\mathbb{C} \setminus f(\mathbb{C})$ has nonempty interior, that is, there is a point $\alpha \in \mathbb{C} \setminus f(\mathbb{C})$ and a $\delta > 0$ so that $D(\alpha, \delta) \subseteq \mathbb{C} \setminus f(\mathbb{C})$. Show that f is constant. (This is a weak version of the so-called "Little Picard Theorem" which states that an entire function assumes all possible complex values with at most a single exception.)