Math 3124A/9024A Assignment 7

University of Western Ontario

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1. (Bak–Newman E.7.1) Show that if f is analytic and non-constant on the closure of a bounded region D, then Re f and Im f assume their maxima and minima on the boundary.

2. For $\alpha \in D(0;1)$, define the function
$$B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}.$$
B_α is analytic on the unit disk. Show that B_α is a bijection of the unit disk $D(0;1)$ onto itself with analytic inverse. What is the inverse of B_α?

3. (Bak–Newman E.7.5) Suppose f is *entire* and $|f| = 1$ on $|z| = 1$. Prove $f(z) = Cz^n$ for some constant C. [*Hint*: First use the maximum and minimum modulus theorem to show
$$f(z) = C\prod_{j=1}^{n} \frac{z-\alpha_j}{1-\bar{\alpha}_j z}.]$$

4. (Bak–Newman E.7.6) Let f be a rational function, that is, $f = P/Q$, where P and Q are polynomials. Further, suppose that the zeroes of Q are contained in the unit disk $D(0;1)$ (the zeroes of Q are known as the "poles" of f). Find another rational function g with no poles in the unit disk and such that $|f(z)| = |g(z)| = 1$ whenever $|z| = 1$.

5. (Bak–Newman E.7.9) Suppose f is analytic in $D(0;2)$ with $|f| \leq 10$ and such that $f(1) = 0$. Find the best possible upper bound for $|f(1/2)|$. Is this upper bound attained by a particular function?

6. (Bak–Newman E.8.2) Prove that every convex region is simply connected.

7. [**MATH 9024 STUDENTS ONLY**] Show that the automorphism group of the unit disk $D(0,1)$ is
$$\left\{ e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z} \;:\; \text{where } \alpha \in D(0,1) \text{ and } \theta \in \mathbb{R} \right\}.$$

That is, suppose f is an analytic bijection of the unit disk onto itself with analytic inverse. Then conclude that f is of the form

$$f(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z}$$

for some constants $\alpha \in D(0, 1)$ and $\theta \in \mathbb{R}$. [*Hint: One way to approach this is to use #2 above and Schwarz' Lemma. Also consider Proposition 3.5 in the book.*]