Math 3124A/9024A Assignment 7

University of Western Ontario

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1. (Bak–Newman E.8.1) A set S is called *star-like* if there exists a point $\alpha \in S$ such that the line segment connecting α and z is contained in S for all $z \in S$. Show that a star-like region is simply connected. [*Hint:* Show that $\gamma : \gamma(t) = tz + (1-t)\alpha$, $t \geq 1$ is contained in the complement for any z in the complement.

2. (Bak–Newman E.8.7) Show that any closed polygonal path can be decomposed into a finite union of simple closed polygonal paths and line segments traversed twice in opposite directions. (For this you may assume the polygonal path consists of only vertical and horizontal segments.)

3. (Bak–Newman E.8.8) Show that $f(z) = \pi i + \int_{-1}^{z} \mathrm{d}\zeta/\zeta$ defines an analytic branch of the logarithm in $\mathbb{C} \setminus \{\mathrm{Re}\ (z) \geq 0\}$ by computing the integral. [*Hint:* As done in class, fix $z \in \mathbb{C} \setminus \mathrm{Re}\ (z) \geq 0\}$ and integrate $1/\zeta$ from -1 to $-|z|$ along a straight line and then from $-|z|$ to z along a circular arc to show that $f(z) = \ln|z| + i\mathrm{Arg}(z)$, where $0 < \mathrm{Im}\ f(z) = \mathrm{Arg}(z) \leq 2\pi$.]

4. (Bak–Newman E.8.9) Define a function f analytic in $\mathbb{C} \setminus \{\mathrm{Re}\ (z) \leq 0\}$ with the property that $f(x) = x^x$ whenever x is a positive real number. Find $f(i)$ and $f(-i)$.

5. Suppose f has an isolated singularity at z_0 and that $f(z) \to \infty$ as $z \to z_0$. Show that f has a pole at z_0.

6. [**MATH 9024 STUDENTS ONLY**] Prove that there is no branch of the logarithm defined on $G = \mathbb{C} \setminus \{0\}$.