Math 3124A/9024A Assignment 9

University of Western Ontario

Fall 2023

1. (Bak–Newman E.9.3) Suppose that f is an entire one-to-one function. Show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$. [*Hint*: If f is a polynomial, then the fundamental theorem of algebra says it must be linear (why?). If it is not a polynoimal, then $g(z) := f(1/z)$ has an essential singularity at $z = 0$ (why?), and the Casorati–Weierstrass Theorem shows that it cannot be one-to-one (why?)]

2. (Bak–Newman E.9.9) Classify the singularities of

   (a) $\dfrac{1}{z^4 + z^2}$

   (b) $\cot z$

   (c) $\csc z$

   (d) $\dfrac{\exp(1/z^2)}{z - 1}$

3. (Bak–Newman E.9.10) Find the Laurent expansion for

   (a) $\dfrac{1}{z^4 + z^2}$ about $z = 0$

   (b) $\dfrac{\exp(1/z^2)}{z - 1}$ about $z = 0$

   (c) $\dfrac{1}{z^2 - 4}$ about $z = 2$

4. [**MATH 9024 STUDENTS ONLY**] (Bak–Newman E.9.14) Show that if f is analytic in $z \neq 0$ and "odd" (i.e. $f(-z) = -f(z)$), then all the even terms in its Laurent expansion about 0 are 0.