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HW 2

Math 3122
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1. (a) $P_y = -6xy + i(3x^2 - 3y^2 - 1)$ and $P_x = 3x^2 - 3y^2 - 1 + i(6xy)$, so $iP_x = i(3x^2 - ey^2 - 1) - 6xy = P_y$, so P is analytic.
   (b) $P_x = 2x$ and $P_y = 2yi$. So $iP_x = 2xi \neq 2yi$. so P is not analytic.
   (c) $P_x = 2y - i2x$ and $P_y = 2x - i2y$. So $iP_x = 2x - i2y = P_y$. So P is analytic.

2. *Proof.* Suppose P is analytic. Then $P_y = iP_x$. But, this is only true when $P_y = P_x = 0$, that is, P is constant. □

3. (a) $P(z) = z^3 - z$, so $P'(z) = 3z^2 - 1 = 3x^2 - 3y^2 - 1 + i(6xy) = P_x$.
   (b) Not analytic.
   (c) $P(z) = -iz^2$, so $P'(z) = -2zi = -2xi + 2y = P_x$.

4. Suppose f, g are differentiable.
   (a) *Proof.*

$$\begin{aligned}(f+g)'(z) &= \lim_{h\to 0} \frac{h_1(z+h) - h_1(z)}{h} \\ &= \lim_{h\to 0} \frac{f(z+h) + g(z+h) - f(z) - g(z)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(z+h) - f(z)}{h} + \frac{g(z+h) - g(z)}{h} \right) \\ &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} + \lim_{h\to 0} \frac{g(z+h) - g(z)}{h} \\ &= f'(z) + g'(z)\end{aligned}$$

□

   (b) *Proof.*

$$\begin{aligned}(fg)'(z) &= \lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z+h)g(z) + f(z+h)g(z) - f(z)g(z)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(z+h)(g(z+h) - g(z))}{h} + \frac{g(z)(f(z+h) - f(z))}{h} \right) \\ &= \lim_{h\to 0} f(z+h)\frac{g(z+h) - g(z)}{h} + \lim_{h\to 0} g(z)\frac{f(z+h) - f(z)}{h} \\ &= f(z)g'(z) + g(z)f'(z)\end{aligned}$$

□

   (c) *Proof.* From the hint, we have $(1/g)' = -\frac{g'(z)}{(g(z))^2}$. So, by the product rule,

$$\begin{aligned}\left(\frac{f}{g}\right)'(z) = \left(f \cdot \frac{1}{g}\right)'(z) &= -\frac{f(z)g'(z)}{(g(z))^2} + \frac{f'(z)}{g(z)} \\ &= \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}\end{aligned}$$

□

5. *Proof.* Let $P_n(z) = \alpha_n z^n$. Then

$$\begin{aligned} P'(z) &= \alpha \lim_{h\to 0} \frac{(z+h)^n - z^n}{z} \\ &= \alpha \lim_{h\to 0} nz^{n-1} + \frac{n(n-1)}{2!} z^{n-2}h + \cdots + h^{n-1} \\ &= \alpha n z^{n-1} \end{aligned}$$

since the terms past nz^{n-1} vanish due to the vanishing h factor. So for any $P(z)$, we have $P(z) = P_0(z) + P_1(z) + \cdots + P_n(z)$, so by the sum rule above, the statement follows. $\square$

6. We have $\limsup_{n\to\infty} |c_n|^{1/n} = 1/R$.

   (a) Since

$$\begin{aligned} \limsup_{n\to\infty} |n^p c_n|^{1/n} &= \lim_{n\to\infty} |n^p|^{1/n} \cdot \limsup_{n\to\infty} |c_n|^{1/n} \\ &= 1 \cdot \limsup_{n\to\infty} |c_n|^{1/n} \\ &= 1/R, \end{aligned}$$

   the radius of convergence is R.

   (b) Since $\limsup_{n\to\infty} ||c_n||^{1/n} = \limsup_{n\to\infty} |c_n|^{1/n} = 1/R$, the radius of convergence is R.

   (c) Note that

$$\limsup_{n\to\infty} a_n^2 = \left(\limsup_{n\to\infty} a_n\right)^2$$

   if $a_n, b_n > 0$. So,

$$\limsup_{n\to\infty} |c_n^2|^{1/n} = \limsup_{n\to\infty} \left(|c_n|^{1/n}\right)^2 = \left(\limsup_{n\to\infty} |c_n|^{1/n}\right)^2 = 1/R^2.$$

   . So the radius of convergence is R^2.

7. (a) *Proof.* Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n > N$, $\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \epsilon$. Then

$$\begin{aligned} |a_n| = \frac{|a_n|}{|a_{n-1}|} \cdots \frac{|a_{N+1}|}{|a_N|} |a_N| &< (L+\epsilon)^{n-N} |a_N| \\ \implies |a_n|^{1/n} &< (L+\epsilon)^{1-N/n} |a_N|^{1/n} \\ \implies \lim_{n\to\infty} |a_n|^{1/n} &\le L + \epsilon \end{aligned}$$

   and since $\epsilon > 0$ is arbitrary, $\lim_{n\to\infty} |a_n|^{1/n} = L$. $\square$

   (b) *Proof.* Let $N \in \mathbb{N}$. For all $n > N$ where n is even, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{-(n+1)}}{2^{-n+2}}\right| = \frac{1}{8}$. Similarily, when n is odd, $\left|\frac{a_{n+1}}{a_n}\right| = 8$. Clearly, the sequence does not converge since it oscillates infinitely between 8 and 1/8, so the limit does not exist.

   We have that $a_n^{1/n} = 1/2$ when n is odd and $a_n^{1/n} = (2^{-n} \cdot 2^2)^{1/n} = \frac{4^{1/n}}{2}$ when n is even, which is monotonically decreasing since $4^{1/n} \ge 4^{\frac{1}{n+1}}$ for all n. So, $\sup\{a_n^{1/n} : n \ge N\} = \max\{1/2, \frac{4^{1/N}}{2}\}$, so $\limsup a_n^{1/n} = \lim_{N\to\infty} \max\{1/2, \frac{4^{1/N}}{2}\} = 1/2$.

   $\square$

8. *Proof.* Suppose that $f(z) = \sum C_n z^n = 1$ for $z = \frac{1}{2}, \frac{1}{3}, \ldots$. Then, since f is continuous, $f(0) = \lim_{z\to 0} f(z) = \lim_{k\to\infty} f(1/k) = 1$. Now define $g = f - 1$. Then g is also continuous, and $g(0) = \lim_{z\to 0} g(z) = \lim_{k\to\infty} f(1/k) - 1 = 0$. So g is zero at all points of a non-zero sequence convergent to zero, so by the Uniqueness Theorem for Power Series, $g \equiv 0$. So $f \equiv 1$ and $f'(0) = 0 \not> 0$. □