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HW 3

Math 3122
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1. *Proof.* Define $\epsilon(\eta) = \frac{f(z)-f(\eta)}{z-\eta} - f'(z)$ such that $\lim_{\eta\to z}\epsilon(z) = 0$ and $\delta(\zeta) = \frac{g(w)-g(\zeta)}{w-\zeta} - g'(w)$ such that $\lim_{\zeta\to w}\delta(\zeta) = 0$. Then, from the definition of δ, setting $w = f(z)$ and $\zeta = f(\eta)$, we have $g(f(z)) - g(f(\eta)) = (\delta(f(\eta)) + g'(f(z)))(f(z) - f(\eta))$. Dividing both sides by $z - \eta$, we have

$$\frac{g(f(z)) - g(f(\eta))}{z-\eta} = (\delta(f(\eta)) + g'(f(z)))\frac{(f(z) - f(\eta))}{z-\eta}$$

when $z \neq \eta$. So,

$$\begin{aligned}\lim_{\eta\to z}\frac{g(f(z)) - g(f(\eta))}{z-\eta} &= \lim_{\eta\to z}(\delta(f(\eta)) + g'(f(z)))\frac{(f(z) - f(\eta))}{z-\eta} \\ &= g'(f(z))f'(z)\end{aligned}$$

where the first equality holds since f is continuous and $\eta \to z$ implies $f(\eta) \to f(z)$, thus $\lim_{\eta\to z}\delta(f(\eta)) = 0$, by definition of δ. □

2. By the Cauchy-Riemann equations, $u_x = 2x = v_y$ and $u_y = -2y = -v_x$. So $v(x, y) = 2xy$. So, all such analytic functions have the form $f = x^2 - y^2 + i(2xy) + C$, where C is any constant.

3. (a) *Proof.* We have $u = e^x \cos y$ and $v = e^x \sin y$, so $u_x = e^x \cos y = v_y$ and $u_y = -e^x \sin y = -v^y$. So e^x is entire. □

   (b) *Proof.*

$$\begin{aligned}e^{z_1+z_2} &= e^{x_1+x_2}(\cos(y_1 + y_2) + i\sin(y_1 + y_2)) \\ &= e^{x_1+x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i \sin y_1 \cos y_2 + i \cos y_1 \sin y_2) \\ &= e^{x_1}(\cos y_1 + i \sin y_1)e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{z_1}e^{z_2}\end{aligned}$$

□

4. (a) $2\sin z\cos z = 2\left(\frac{1}{2i}(e^{iz} - e^{-iz})\frac{1}{2}(e^{iz} + e^{-iz})\right) = \frac{1}{2i}(e^{2iz} - e^{-2iz}) = \sin 2z$.

   (b) $\sin^2 z + \cos^2 z = -\frac{1}{4}(e^{iz} - e^{-iz})^2 + \frac{1}{4}(e^{iz} + e^{-iz})^2 = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz} - (e^{2iz} - 2 + e^{-2iz})) = 1$.

   (c) $(\sin z)' = \left(\frac{1}{2i}(e^{iz} - e^{-iz})\right)' = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$.