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HW 5

Math 3124
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1. First we prove the following lemma.

Lemma 1. *Let $\{z_n\} \subseteq \mathbb{C}$ be infinite and bounded. Then there exists a subsequence $\{z_{n_k}\} \subseteq \{z_n\}$ convergent to some point $z \in \mathbb{C}$.*

Proof. Let $\{z_n\}$ be bounded. It suffices to show that a subsequence of $\{z_n\}$ where both the real and imaginary parts converge. Since $\{z_n\}$ is bounded, $\{\Re(z_n)\}$ is bounded, so by the lemma, there exists a subsequence $\{z_{n_k}\}$ such that the real parts converge. Since $\{z_{n_k}\} \subseteq \{z_n\}$, it is also bounded, so again by Bolzano-Weierstrass $\{\Im(z_{n_k})\}$ is bounded and there exists a subsequence $\{z_{n_{k_l}}\} \subseteq \{z_{n_k}\}$ such that the imaginary parts converge. Thus the real and imaginary parts of $\{z_{n_{k_l}}\}$ converge, so the subsequence is convergent. □

Now we prove the theorem.

Proof. Let f be a continuous complex-valued function on a compact set $K \subseteq \mathbb{C}$. Set $M := \sup_{z \in K} |f(z)|$. Consider the sequence $\{f(z_n)\}$ defined such that $M - \frac{1}{n} < |f(z_n)|$ for all $n \geq 1$, which exists by definition of supremum. Since $|f(z_n)| \leq M$ for all $z_n \in K$, the sequence converges to an α with $|\alpha| = M$. Since K is compact, it is bounded, so by Bolzano-Weierstrass, there exists a subsequence $\{z_{n_k}\} \subseteq \{z_n\}$ convergent to some point $z \in \mathbb{C}$. Since f is continuous, $\{f(z_{n_k})\}$ also converges, and since $\{f(z_{n_k})\} \subseteq \{f(z_n)\}$, it converges to α. Finally, since K is compact and z is a limit point of K, $z \in K$. Thus $\alpha = f(z)$.

Note that in the above, we assumed M was finite. This can be shown as follows. Suppose otherwise. Then $f(K)$ is unbounded. But this contradicts that f is continuous, since continuous functions map compact sets to compact sets. □

2. *Proof.* Let f be entire. Recall that in the power series expansion $f(z) = \sum C_k z^k$, $C_k = \frac{f^{(k)}(0)}{k!}$. From the proof of Theorem 5.5, we found that $C_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}} dw$. where C is a circle centered at the origin of radius $|w|$ containing z. Setting the two equal, we have $f^{(k)}(0) = \frac{k!}{2\pi i} \int_C \frac{f(w)}{w^{k+1}} dw$ for $k = 1, 2, \ldots$. Define $g(z) := f(z + a)$. Then

$$\begin{aligned} f^{(k)}(a) = g^{(k)}(0) &= \frac{k!}{2\pi i} \int_C \frac{g(\omega)}{\omega^{k+1}} d\omega \\ &= \frac{k!}{2\pi i} \int_C \frac{f(\omega + a)}{\omega^{k+1}} d\omega \\ &= \frac{k!}{2\pi i} \int_C \frac{f(w)}{(w-a)^{k+1}} dw \end{aligned}$$

using the parameterization $\omega = w - a$. □

3. (a) *Proof.* Suppose f is entire with $|f| \leq M$ along $|z| = R$. From above, we have that the coefficients of the power series expansion of f about 0 is given by $C_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}} dw$, where C is a circle centered at the origin of radius $|w|$ containing z. Then, the integrand is bounded above by $\frac{M}{R^{k+1}}$. The length of C is $2\pi R$, so by the M-L Theorem, $|C_k| \leq |\frac{1}{2\pi i} \left(\frac{M}{R^{k+1}}\right)(2\pi R)| = \frac{M}{R^k}$. □

   (b) *Proof.* Polynomial functions are entire, so by above, with $M = 1$ and $R = 1$, $|C_k| \leq \frac{M}{R^k} = 1$. □

4. *Proof.* Let f be entire with $|f(z)| \leq A + B|z|^k$. Then, along $|z| = R$, $|f| \leq A + BR^k$. From above, $|C_j| \leq \frac{A+BR^k}{R^j}$. Now suppose $j > k$. Taking circles of radius R of arbitrary size, $\lim_{R\to\infty} \frac{A+BR^k}{R^j} = 0$, since $j > k$. □

5. *Proof.* Let f be entire with $|f(z)| \leq A + B|z|^{3/2}$. From above, $|C_k| = 0$ for $k \geq 3/2$. That is, $|C_k| \neq 0$ only for $k = 0, 1$. So $f(z) = C_0 + C_1 z$. □

6. *Proof.* Suppose f is entire. Since it is entire, it is continuous, and thus bounded on the compact set $0 \leq x, y \leq 1$. By periodicity, it is also bounded on all 1-by-1 squares $a \leq x, y \leq a + 1$ where $a \in \mathbb{Z}$. So, the function is bounded on the entire complex plane, so it is constant, by Liouville's Theorem. □

7. *Proof.* ($\rightarrow$) Suppose $P(z) = (z - \alpha)^k Q(z)$ where Q is analytic and $Q(\alpha) \neq 0$. Then

$$P'(z) = k(z-\alpha)^{k-1}Q(z) + (z-\alpha)^k Q'(z) = (z-\alpha)^{k-1}(kQ(z) + (z-\alpha)Q'(z))$$

So $P'(z) = (z - \alpha)^{k-1}Q_1(z)$ with $Q_1(\alpha) \neq 0$. Repeating as in the first part of the proof, $P^{(n)}(z) = (z - \alpha)^{k-n}Q_n(z)$ with $Q_n(\alpha) \neq 0$. Note that $P^{(k)}(z) = k!Q_k(z) \neq 0$ since $Q_k(z) \neq 0$. So $P^{(0)}(\alpha) = P^{(1)}(\alpha) = \cdots = P^{(k-1)}(\alpha) = 0$, and $P^{(k)}(\alpha) \neq 0$.

($\leftarrow$) Suppose $P(\alpha) = P'(\alpha) = \cdots = P^{(k-1)}(\alpha) = 0$ and $P^{(k)}(\alpha) \neq 0$. We show that $P(z) = (z - \alpha)^k Q(z)$ where Q is analytic and $Q(\alpha) \neq 0$. First note that since $P^{(k-1)}(\alpha) = 0$, $P^{(k-1)}(z) = (z - \alpha)Q(z)$ by the Factor Theorem. Also, $Q(\alpha) \neq 0$ since

$$P^{(k)}(z) = Q(z) + (z - \alpha)Q'(z)$$

and $P^{(k)}(z) \neq 0$ by assumption. We finish with induction. Suppose $P^{(j)}(z) = (z - \alpha)^n Q(z)$ with Q analytic and $Q(\alpha) \neq 0$, and $j < k$ and $n \geq 1$. We show that $P^{(j-1)}(z)$ has root α with multiplicity $n + 1$. From the initial assumption, $P^{(j-1)}(z) = (z - \alpha)S(z)$ with S analytic. Differentiating gives

$$P^{(j)}(z) = (z - \alpha)^{n+1}Q(z) = S(z) + (z - \alpha)S'(z),$$

by the inductive hypothesis. so $S(z)$ has multiplicity n. Thus $P^{(j-1)}(z) = (z - \alpha)S(z)$, has multiplicity $n + 1$.

We have that $P^{(k-1)}(z)$ has root α with multiplicity 1, so $P^{(0)}(z)$ has root α with multiplicity $1 + k - 1 = k$, as desired. □