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HW 7

Math 3124
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1. *Proof.* Let f be analytic and non-constant on the closure of a bounded region D. Suppose for contradiction that $\operatorname{Re} f$ is maximum at an interior point $z_0 \in D$. By the open mapping theorem, $f(D)$ is open, so $f(z_0) + \epsilon \in f(D)$ for some positive real ϵ. But then $\operatorname{Re}(f(z_0) + \epsilon) > \operatorname{Re}(f(z_0))$, contradicting that $\operatorname{Re} f$ is maximum at z_0.

   Similarily, if $\operatorname{Re} f$ is minimum at an interior point $z_0 \in D$, then $f(z_0) - \epsilon \in f(D)$ for some positive real ϵ, and $\operatorname{Re}(f(z_0) - \epsilon) < \operatorname{Re}(f(z_0))$, contradicting that $\operatorname{Re} f$ is minimum at z_0.

   Analogously, $\operatorname{Im} f$ can't be maximum or minimum at an interior point $z_0 \in D$, for then $f(z_0) \pm \epsilon \in f(D)$ for some imaginary ϵ with positive modulus would be smaller than the supposed minimum or larger than the supposed maximum.

   Thus, $\operatorname{Re} f$ and $\operatorname{Im} f$ must both attain their maximum and minimum on the boundary of D. □

2. First we prove the following lemma.

   Lemma 1. *Let $f : S \to T$ be a non-constant analytic function on its domain. If $f(z) \in T$ is a boundary point of T, then z is a boundary point of S.*

   Proof. Suppose otherwise: that $f(z)$ is a boundary point of T but z not a boundary point of S. Then z is an interior point of S, so there exists a disk $D(z; r) \subset S$. By the open mapping theorem, $f(D(z; r))$ is open, so $f(z)$ is an interior point of T, contradicting that $f(z)$ is a boundary point of T. □

   Now we prove the theorem.

   Proof. We have that $B_\alpha : D(0; 1) \to A$ is analytic in its domain. We show that A is the unit disk, and B_α is a bijection. First, note that if $|z| = 1$, then $|B_\alpha(z)| = 1$. So,by the Maximum-Modulus theorem, since B_α is non-constant, (e.g. $B_\alpha(0) = -\alpha \neq 1$), there is some $|B_\alpha(z)| < 1$ for some $z \in \operatorname{Int} D(0; 1)$. Let $\alpha \in A$ be one such value, with $|\alpha| < 1$. Consider an arbitrary chord T of $C(0; 1)$ passing through α. Let $X = T \setminus C(0; 1)$ (that is, T excluding its endpoints). Then, $X \subseteq A$, since if it weren't, then there would be some $f(z) \in X$ that is a boundary point of A. But, this isn't possible, since by the Lemma, z would be a boundary point with $|z| = 1$. But then $|f(z)| = 1$, contradicting that $f(z)$ is on a chord in the interior of $D(0; 1)$. So $X \subseteq A$. Thus, A contains all chords (minus their endpoints) passing through α, so A is the interior of the unit disk. Finally, since f is continuous and $D(0; 1)$ is compact, A is compact. So A is the unit disk.

   Now we show that B_α is a bijection by showing that it, composed with its inverse, is an identity map in $D(0; 1)$. Define its inverse $B_\alpha^{-1} : A \to D(0; 1)$ by

$$B_\alpha^{-1}(\beta) = \frac{\beta + \alpha}{1 + \overline{\alpha}\beta}$$

   Then

$$\begin{aligned}(B_\alpha^{-1} \circ B_\alpha)(z) &= \frac{\frac{z-\alpha}{1-\overline{\alpha}z} + \alpha}{1 + \overline{\alpha}\frac{z+\alpha}{1+\overline{\alpha}z}} \\ &= \frac{z - \alpha\bar{\alpha}z}{1 - \alpha\bar{\alpha}} \\ &= z\end{aligned}$$

Finally, note that this inverse is analytic on the unit disk, since it is a rational functions whose denominator is non-zero within the unit disk, since $1 + \overline{\alpha} z = \overline{\alpha}(1/\overline{\alpha} + z)$, and $|1/\overline{\alpha}| > 1$ since $|\alpha| < 1$, and $|z| \leq 1$. □

3. *Proof.* First, note that f has finitely many zeroes inside the unit disk, for if it didn't, then by compactness of the unit disk, there would be a sequence of zeroes convergent to a point within the domain of analycity of f. Then by the Uniqueness Theorem, $f \equiv 0$; contradicting that $|f| = 1$ on $|z| = 1$. So let $\alpha_1, \ldots, \alpha_n$ be the finitely many zeros of f. Then

$$g(z) = \frac{f(z)}{\prod_{j=1}^{n} \frac{z-\alpha_j}{1-\overline{\alpha_j} z}}$$

is non-zero at all points inside the unit disk, and $|f| = 1$ on the unit disk boundary. So by the Maximum-Modulus and Minimum-Modulus theorems, g is constant. So

$$f(z) = C \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z}$$

Finally, since f is entire, $\alpha_1 = \cdots = \alpha_n = 0$, so $f(z) = Cz^n$. □

4. Let $\alpha_1, \ldots, \alpha_n$ be the zeroes of Q. Define

$$g(z) = f(z) \prod_{j=1}^{n} B_{\alpha_j}(z)$$

then $B_{\alpha_j}(z) = 0$ when $z = \alpha_1, \ldots, \alpha_n$, so g has no poles within the unit disk. And since $|B_\alpha(z)| = 1$ when $|z| = 1$ (from above), $|f(z)| = |g(z)|$ when $|z| = 1$.

5. Let $g(z) = \frac{1}{10} f(2z)$ so that the image of g is contained in the unit disk and $g(1/2) = 0$.

$$g(z) \ll B_{1/2}(z) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2} z}$$

So $g(1/4) \ll 2/7$. So $f(1/2) \ll 20/7$. This upper bound is attained by $f(z) = 10B_{1/2}(z/2)$ since $f(1/2) = 10B_{1/2}(1/4) = 20/7$.

6. *Proof.* We prove the contrapositive. Suppose that a region D is not simply connected. Then there is a point in its complement $z \in \tilde{D}$ such that every path connecting z to ∞ has some point on the path $\gamma(t)$ with $d(\gamma(t), \tilde{D}) > \epsilon$ for some $\epsilon > 0$. Consider the straight line paths connecting z to ∞, $\alpha(t) = z + t$ and $\beta(t) = z - t$. Choose t_α and t_β such that $\alpha(t_\alpha) = a$ and $\beta(t_\beta) = b$ are epsilon away from $\tilde{D}$. Then $a, b \in D$, but the straight line path L connecting a and b is not entirely contained within D, since $z \in L$. So D is not convex. □