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HW 8

1. *Proof.* Let $z \in \tilde{S}$. Consider $\gamma(t) = tz + (1-t)\alpha$ for $t \geq 1$. Then γ connects z to infinity and is contained in $\tilde{S}$, since if it weren't, there would be a point $z' = \gamma(t')$ in S where $t' \geq 1$. But then z' is connected to α by the line segment $\gamma(t)$, $0 \leq t \leq t'$, which is not completely contained in S since $z = t(1) \in \tilde{S}$, contradicting the fact that S is star-like. □

2. *Proof.* Let $\gamma : \gamma(t), a \leq t \leq b$ be a closed polygonal path. If it is simple we are done, so suppose otherwise. Then $\gamma(t_1) = \gamma(t_2)$ for some $t_1 \neq t_2$ (possibly many) that are not the endpoints. Either this is a single intersection point, that is, the intersecting line segments are secant or touch corners, or it is a neighborhood of intersection points, that is, the intersection is two colinear line segments. We show that we can decompose γ into closed polygonal paths and line segments traversed twice in opposite directions without the given intersection(s), from which the claim follows by induction by applying the same argument to each of the new closed polygonal paths until no intersections remain.

   If this is a single intersection point, that is, $\gamma(t_1) = \gamma(t_2)$ for $t_1 \neq t_2$ but $\gamma(t_1') \neq \gamma(t_2')$ for all $t_1' \in (t_1 - \epsilon, t_1 + \epsilon) \setminus t_1$ and $t_2' \in (t_2 - \delta, t_2 + \delta) \setminus t_2$ for all $\epsilon, \delta > 0$, then $\gamma = \gamma_1 \cup \gamma_2$ where $\gamma_1 = \gamma(t), t \in [a, t_1] \cup [t_2, b]$ and $\gamma_2 = \gamma(t), t \in [t_1, t_2]$.

   If instead it is a neighborhood of intersection points, then $\gamma(t_1) = \gamma(t_2)$ for some neighborhoods T_1 and T_2 around and including t_1 and t_2. Define $x := \inf T_1$, $y := \sup T_1$, $c := \inf T_2, d := \sup T_2$. We consider two cases: where the line segments travel in the same direction and opposite directions.

   If the line segments travel in the same direction, then $x = c$ and $y = d$, so we can decompose γ as $\gamma_1 \cup \gamma_2$, where $\gamma_1 = \gamma(t), t \in [x, y] \cup [d, c]$ and $\gamma_2 = \gamma(t), t \in [a, x] \cup [c, y] \cup [d, b]$. Note that γ_1 is a line segment traversed twice in opposite directions, and γ_2 is a closed polygonal path.

   If the line segments travel in opposite directions, then $x = d$ and $y = c$. Decompose $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where $\gamma_1 = \gamma(t), t \in [a, d] \cup [d, b]$, $\gamma_2 = \gamma(t), t \in [y, c]$ and $\gamma_3 = \gamma(t), t \in [x, y] \cup [c, d]$. Note that γ_1 and γ_2 are closed polygonal paths, and γ_3 is a line segment traversed in two directions. □

3. *Proof.* We have $\int_{-1}^{z} d\zeta/\zeta = \int_{-1}^{-|z|} d\zeta/\zeta + \int_{-|z|}^{z} d\zeta/\zeta$. The first term is

$$\int_{-1}^{-|z|} d\zeta/\zeta = \ln|\zeta|\Big|_{-1}^{-|z|} = \ln|-|z|| - \ln|-1| = \ln|z| - \ln 1 = \ln|z|$$

   Integrating from $|z|$ to z along $C : \zeta(\theta) = |z|e^{i(\theta-\pi)}, 0 \leq \theta \leq \operatorname{Arg} z + \pi$,

$$\int_C d\zeta/\zeta = \int_0^{\operatorname{Arg} z+\pi} \frac{\zeta'(\theta)}{\zeta(\theta)} d\theta = \int_0^{\operatorname{Arg} z+\pi} \frac{i|z|e^{i(\theta-\pi)}}{|z|e^{i(\theta-\pi)}} d\theta = \int_0^{\operatorname{Arg} z+\pi} i d\theta = i(\operatorname{Arg} z + \pi)$$

   So $f(z) = \pi i + \int_{-1}^{z} d\zeta/\zeta = \ln|z| + i \operatorname{Arg}(z) + i2\pi$, so $f(z)$ is an analytic branch of $\ln z$. □

4. *Proof.* Note that $f(x) = x^x = e^{x \ln x}$. From class, we showed $\ln z = \ln|z| + i \operatorname{Arg} z$ for z with positive real part. So

$$f(z) = \exp(z \ln z) = \exp(z \ln|z| + iz \operatorname{Arg} z) = \exp(z \ln|z|) \exp(iz \operatorname{Arg} z)$$

   Since $\ln z$ is analytic on this domain, $f(z)$ is analytic on this domain. Note that when $z \in \mathbb{R}$, the second factor is 1, so $f(x) = x^x$ is real-valued on $\mathbb{R}$. We have

$$f(i) = \exp(i \ln 1 + i^2\pi/2) = \exp(-\pi/2) = \frac{1}{e^{\pi/2}}$$

and

$$f(-i) = \exp(-i \ln 1 + i^2\pi/2) = \exp(-\pi/2) = \frac{1}{e^{\pi/2}}$$

□

5. *Proof.* Since $f(z) \to \infty$ as $z \to z_0$, f is unbounded in the deleted neighborhood of z_0. So z_0 is not a removable singularity. Also, since $f(z) \to \infty$ as $z \to z_0$, we can choose some deleted δ-neighborhood X of z_0 such that $|f(z)| > 1$ for all $z \in X$. So, $D(0; \frac{1}{2}) \subseteq X \setminus \mathbb{C}$ is disjoint from X. So X is not dense in $\mathbb{C}$. So by the Caserati-Weierstrass theorem, z_0 is not an essential singularity. z_0 is neither an essential singularity nor a removable singularity, so it must be a pole. □