David Tran
251169871

HW 9

Math 3124
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1. *Proof.* Suppose f is an entire one-to-one function. If f is a polynomial, then by the Fundamental Theorem of Algebra, it has n zeros (counting multiplicity), where n is the degree of f. Since f is one-to-one, it has at most 1 zero of degree 1. Thus, $n = 1$, so f is a linear function. It remains to show that f must be a polynomial.

   Suppose $f(z)$ is not a polynomial. Then since f is entire, its Taylor expansion at $z = 0$ converges everywhere and has infinitely many terms. So, the principal part of the Laurent series of $f(1/z)$ has infinitely many negative terms. Note that these expansions must be equivalent by the Uniqueness of Laurent expansions. Thus, $f(1/z)$ has an essential singularity at $z = 0$. By the Casorati-Weierstrass Theorem, $f(1/z)$ maps the deleted neighborhood D of $z = 0$ to a dense subset $f(D) \subseteq \mathbb{C}$. Choose an open set disjoint from D, say U. By the density of $f(D)$ in $\mathbb{C}$, there is an element $x \in f(D)$ arbitrarily close to an element in $f(U)$. Since $f(U)$ is open by the open mapping theorem, $x \in f(U)$. So $f(D)$ and $f(U)$ are not disjoint despite D and U being disjoint, so f is not injective; a contradiction. So f is a polynomial. □

2. (a) $z = 0$ and $z = \pm i$ are poles since 1 and $z^4 + z^2$ are polynomials and thus entire, and $z^4 + z^2 = 0$ at these values. $z = 0$ is a pole of order 2 since it is a zero of order 2 of $z^4 + z^2 = z^2(z^2 + 1)$.

   (b) $z = k\pi$ for $k \in \mathbb{Z}$ are poles since $\cot z = \cos z / \sin z$ and $\sin z = 0$ at these values. And, $\sin z$ and $\cos z$ are entire.

   (c) $z = k\pi$ for $k \in \mathbb{Z}$ are poles since $\csc z = 1/\sin z$ and $\sin z = 0$ at these values. And, $\sin z$ and 1 are entire.

   (d) $z = 1$ is a pole since $z - 1 = 0$ there and $z - 1$ and $\exp(1/z^2)$ are analytic at $z = 1$. $z = 0$ is an essential singularity since it is not a pole since $\exp(1/z^2)$ is analytic at $z = 0$. And, $z = 0$ is not a removable singularity since $\exp(1/z^2)$ is an essential singularity (its Laurent expansion has infitely many terms in its principal part).

3. (a)

$$\frac{1}{z^4 + z^2} = \frac{1}{z^2(z^2+1)} = \frac{1}{z^2}\sum_{k=0}^{\infty}(-1)^k z^{2k} = \sum_{k=0}^{\infty}(-1)^k z^{2k-2}$$

   (b)

$$\frac{\exp(1/z^2)}{z-1} = -\frac{\exp(1/z^2)}{1-z} = -\sum_{k=0}^{\infty} z^k \cdot \sum_{k=0}^{\infty}\frac{1}{k!z^{2k}} = -\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{z^{j-2k}}{k!}$$

   (c) Note that

$$\begin{aligned}\frac{1}{z+2} &= \frac{1}{z-2}\cdot\frac{1}{1+\frac{4}{z-2}} \\ &= \frac{1}{z-2}\cdot\sum_{k=0}^{\infty}\left(-\frac{4}{z-2}\right)^k \\ &= \frac{1}{z-2} + \sum_{k=1}^{\infty}\frac{(-4)^k}{(z-2)^{k+1}}\end{aligned}$$

so

$$
\begin{aligned}
\frac{1}{z^2+4} &= \frac{1}{4}\left(\frac{1}{z-2} - \frac{1}{z+2}\right) \\
&= -\frac{1}{4}\sum_{k=1}^{\infty} \frac{(-4)^k}{(z-2)^{k+1}} \\
&= \frac{1}{4}\sum_{k=2}^{\infty} \frac{(-4)^k}{(z-2)^k}
\end{aligned}
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