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HW 2

1. (a) *Proof.* Suppose it were. Then let $\langle A\rangle = \mathbb{Q}$ with $A \subseteq \mathbb{Q}$ where A is finite. Let d be the product of every denominator in A. Then any linear combination of the rationals in a can be written as x/d for some term x. Now choose a prime p that does not divide d. Then $1/p \in \mathbb{Q}$ can not be generated by A, since x/d is reducible to $1/p$ if and only if p divides d, which it doesn't by construction; a contradiction. □

   (b) *Proof.* Consider $A := \{\frac{a}{2^n} : n \in \mathbb{N}, a \in \mathbb{Z}\}$. It is a subgroup of $\mathbb{Q}$ since $A \subseteq \mathbb{Q}$ and $a/2^n - b/2^n = (a-b)/2^n \in A$, and it is proper since $1/3 \in \mathbb{Q} - A$. It is not cyclic since if it were generated by a single rational, say $r \in A$, then $r/2 \in A$ but $ar \neq r/2$ for $a \in \mathbb{Z}$. □

   (c) *Proof.* Let $a/b \in \mathbb{Q}^+$. By closure, it suffices to show that a and $1/b$ are generated by the set of interest A. Let $a = p_1^{e_1} \dots p_n^{e_n}$ be the prime factorization of a. Then $a = (p_1^{-1})^{-e_1} \dots (p_n^{-1})^{-e_n}$, so $a \in \langle A\rangle$. Similarily, let $b = q_1^{f_1} \dots q_m^{f_m}$ be the prime factorization of b, then $1/b = ((q_1)^{-1})^{-f_1} \dots (q_m^{-1})^{-f_m}$, so $1/b \in \langle A\rangle$. □

2. (a) The centralizers for $\langle i\rangle, \langle j\rangle$, and $\langle k\rangle$ are at least the subgroups themselves, since each are cyclic and thus Abelian. The only greater subgroup is Q_8 itself, which is not Abelian and thus cannot be part of the centralizer. So the centralizers for $\langle i\rangle, \langle j\rangle, \langle k\rangle$ are the subgroups themselves. The centralizer for $\langle 1\rangle$ and $\langle -1\rangle$ is Q_8, since 1 and -1 commute with all of Q_8. Finally, the centralizer for Q_8 is $\{1, -1\}$, since neither i, j, nor k commutes with j, k, nor i, respectively and ny the lattice diagram, there are no greater subgroups for Q_8 that could be the centralizer.

   Since the centralizer is a subgroup of the normalizer, the normalizer of $\langle 1\rangle$ and $\langle -1\rangle$ is Q_8. For the normalizer of $\langle i\rangle$, note that $j\langle i\rangle j^{-1} = \{1, i, -1, -i\} = \langle i\rangle$. So, $j \in N_{Q_8}(\langle i\rangle)$, and since $C_{Q_8}(\langle i\rangle) = \langle i\rangle$, the normalizer must be Q_8. Symmetrically, the normalizer for $\langle j\rangle$ and $\langle k\rangle$ is Q_8. Finally, every group is its own normalizer, so $N_{Q_8}(Q_8) = Q_8$.

   (b) From above, since the normalizer of every subgroup is Q_8, every subgroup is normal.

   For the isomorphism type of the quotient of each subgroup: we have $|Q_8 : \langle -1\rangle| = 4$, so $Q_8/\langle -1\rangle$ is congruent to either V_4 or $\mathbb{Z}_4$. But, for any $g \in Q_8/\langle -1\rangle$, $g^2\langle -1\rangle = \langle -1\rangle$, so the elements $i\langle -1\rangle, j\langle -1\rangle, k\langle -1\rangle$ have order 2, so $Q_8/\langle -1\rangle$ cannot be congruent to $\mathbb{Z}_4$. Thus $Q_8/\langle -1\rangle \cong V_4$.

   Next, we have $|Q_8 : \langle i\rangle| = 2$. Thus $Q/\langle i\rangle \cong \mathbb{Z}_2$. Similarly, $Q/\langle j\rangle \cong Q/\langle k\rangle \cong \mathbb{Z}_2$.

3. (a) *Proof.* Let A be divisible with $B \leq A$. Let $aB \in A/B$ with $a \in A$. Since $x^n = a$ for some non-zero integer n and $x \in A$, $aB = (x^n)B = (xB)^n$. So A/B is divisible. □

   (b)
   - Finite Abelian groups are not necessarily divisible: consider $\mathbb{Z}_2$ under addition. There is no element $y \in \mathbb{Z}_2$ such that $2y = 1$, since $2(0) = 2(1) = 0$.
   - $\mathbb{Z}$ is not divisible: there is no element $y \in \mathbb{Z}$ such that $2y = 1$ since the product of an even integer with any integer is even.
   - $\mathbb{Q}$ is divisible since for any $r \in \mathbb{Q}$ and $n \in \mathbb{N}$, $r/n \in \mathbb{Q}$ and $(r/n)n = r$.
   - $\mathbb{Q}/\mathbb{Z}$ is divisible: let $r\mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ and $n \in \mathbb{N}$. Then $(r/n)\mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ and $(r/n)\mathbb{Z} \cdot n\mathbb{Z} = r\mathbb{Z}$.

4. (a) *Proof.* If A is Abelian, then $A \times A$ is Abelian since $(x, y)(u, v) = (xu, yv) = (ux, vy) = (u, v)(x, y)$ for $x, y, u, v \in A$. The subgroup of any Abelian group is normal, so $D \leq A \times A$ is normal. □

   (b) *Proof.* Let $\sigma = (1\ 2\ 3)$, $\tau = (1\ 2)$, and $\rho = (2\ 3)$, with $\sigma, \tau, \rho \in S_3$. Then $(\rho, \rho) \in D$ and $(\sigma, \tau) \in S_3 \times S_3$, but

$$(\sigma, \tau)^{-1}(\rho, \rho)(\sigma, \tau) = (\tau, \sigma^{-1}) \notin D.$$

so D is not normal. □

5. (a) *Proof.* First we show that for any $g \in G$, $g = x^n z$ where $z \in Z(G), n \in \mathbb{Z}$, and x is such that $G/Z(G) = \langle xZ(G) \rangle$. Let $g \in G$. Then $gZ(G) = x^n Z(G)$ for some m, so $x^{-n}g \in Z(G)$. So there is some $z \in Z(G)$ with $x^{-n}g = z$, that is, $g = x^n z$.

   Now let $g, h \in G$. From above, $gh = (x^n z_1)(x^m z_2) = (x^m z_2)(x^n z_1) = hg$, using commutativity from the fact that $z_1, z_2 \in Z(G)$. □

   (b) *Proof.* By Lagrange's theorem, the order of $Z(G)$ is either pq, p, q or 1. If the order is pq, then $Z(G) = G$ so G is Abelian. If it is p (or q), then $|G/Z(G)| = q$ (or p), so it is cyclic and thus Abelian by (a). To see that $G/Z(G)$ is cyclic, note that for some non-identity element $x \in Z(G)$, $|x| = q$ (or p) by Lagrange's theorem and the fact that q (or p) is prime. So, $|\langle x \rangle| = |Z(G)|$, so $\langle x \rangle = Z(G)$.

   Thus, G is either Abelian or $|Z(G)| = 1$ so $Z(G) = 1$. □

6. *Proof.* Since $xH = Hy$ and $1 \in H$, choose $h \in H$ such that $x(1) = x = hy$. Thus $xy^{-1} = h \in H$, so $xy^{-1} \in H$, which implies $Hx = Hy$. Hence $xH = Hy = Hx$. Finally, since $xH = Hx$, $xHx^{-1} = H$, so $x \in N(H)$. □

7. *Proof.* Since G is finite and $H, N \leq G$, $|HN| = \frac{|H||N|}{|H \cap N|}$. Also, from Corollary 15 in Chapter 3, since $N \trianglelefteq G$, $HN \leq G$. And, $|G : N| = |G|/|N|$ implies $|N| = |G|/|G : N|$. So,

$$
\begin{aligned}
|G| &= \frac{|H||N|}{|H \cap N|}|G : HN| \\
&= \frac{|H||G|}{|H \cap N||G : N|}|G : HN|
\end{aligned}
$$

   Dividing on both sides by $|G|$ gives $|H||G : HN| = |H \cap N||G : N|$. Finally, since $(|H|, |G : N|) = 1$, we must have $|H| = |H \cap N|$ and $|G : HN| = |G : N|$. So $H \leq N$. □

8. *Proof.* Since $|H||G : M| = p$, either $(|H|, |G : M|) = 1$, or $(|H|, |G : M|) = p$. From 7, if the first is true then $H \leq M$. So suppose the second is true. We have that HM is a subgroup and again from 7, $|H||G : HM| = |H \cap M||G : M|$. Also, HM is a subgroup and $M \leq HM$. Since

$$
\begin{aligned}
|HM| &= \frac{|H||K|}{|H \cap K|} \\
&= |H : H \cap K||K|
\end{aligned}
$$

   we have $|HM = k|K|$ for some positive integer k, so either $|G : HM| = p$ or $|G : HM| = 1$. If it is p then from 7, $|H = |H \cap M|$, so $H \leq M$. If instead it is 1, then $|HM| = |G|$, so $HM = G$ and $|HM| = p|M|$. So $|HM| = |H : H \cap M||K|$ and $|H : H \cap M| = p$. □

9. *Proof.* Define $\phi : G \to G/M \times G/N, a \mapsto (aN, aM)$. First we show that ϕ is well-defined. Suppose $g_1 = g_2 \in G/(M \cap N)$, so $g_1 = g_2 m$ for some $m \in M \cap N$. Then $(g_1 M, g_1 N) = (g_2 M, g_2 N)$ so $\phi(g_1) = \phi(g_2)$.

   Next we show that ϕ is a homomorphism. Let $a, b \in G/(M \cap N)$. Then,

$$
\phi(ab) = (abN, abM) = (aM, aN)(bM, bN) = \phi(a)\phi(b)
$$

.

For surjectivity, let $(aN, bM) \in (G/N) \times (G/M)$. Since $G = MN$, $a = m_1n_1$ and $b = m_2n_2$ for some $m_1, m_2 \in M$ and $n_1, n_2 \in N$. So $aM = m_1n_1M = m_1Mn_1 = n_1M$, so $gM = Mg$ where $g = mn$. Similarily, $bN = m_2N$. Thus $\phi(m_2n_1) = (n_1M, m_2N) = (aM, bN)$.

Finally, note that $\phi(a) = (1M, 1N)$, with $a \in M \cap N$. So, $\text{Ker}(\phi) = M \cap N$. Thus, by the First Isomorphism Theorem, since $\phi : G \to G/M \times G/N$ is a surjective homomorphism, $G/(M \cap N) \cong G/M \times G/N$. □