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HW 4

Math 3120
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1. (a) The conjugacy classes of Q_8 are $\{1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$.
   (b) The conjugacy classes of A_4 are identified by their cycle type:

$$\{1\}$$
$$\{(12)(34), (13)(24), (14)(23)\}$$
$$\{(123), (341), (243), (421), \}$$
$$\{(132), (412), (234), (314)\}$$

2. (a) *Proof.* Recall that by Proposition 4.6 the size of the conjugacy class of x is the index of its normalizer. $Z(G) \leq C_G(x)$ so $|G : C_G(x)| \leq |G : Z(G)| = n$. □
   (b) *Proof.* Let G be a group with exactly two conjugacy classes. Since $\{1\}$ forms its own conjugacy class, let $\mathcal{C}$ be the other distinct conjugacy class. By the class equation, $|G| = 1 + |\mathcal{C}|$. Since $|\mathcal{C}| \mid |G|$, we must have $|\mathcal{C}| = 1$. So $|G| = 2$. Thus, only the groups of cyclic order 2 have exactly two conjugacy classes. □

3. *Proof.* We note that an element x is in the center of a group G if and only if the order of its conjugacy class is 1, since $gx = xg \implies gxg^{-1} = x$. It suffices to show then that every non-identity element of S_n has a conjugacy class of order greater than 1. Since in S_n for $n \geq 3$, there is more than one distinct m-cycle for $m \leq n$, for any $\sigma \in S_n$ with cycle decomposition $\sigma = \tau_1 \ldots \tau_n$, we can find a distinct $\sigma' \in S_n$ with the same cycle type (and thus in the same conjugacy class) as σ by choosing a distinct cycle of the same length for each cycle $\tau_1, \ldots, \tau_n$. □

4. *Proof.* Since H is normal, $N_G(H) = G$. So by Corollary 15, $G/C_G(H)$ is isomorphic to some subgroup of $\mathrm{Aut}(H)$. By Proposition 17, $\mathrm{Aut}(H) = |H| - 1 = 6$ since $|H|$ is prime, so $|G/C_G(H)| \mid 6$. But, 7 is the smallest prime dividing $|G| = 203$, so $|G/C_G(H)| = 1$, thus $G = C_G(H)$. So $H \leq Z(G)$. If $H < Z(G)$, then $Z(G) = G$ so G is abelian. If instead $H = Z(G)$, then G/H is cyclic, so G is abelian. □

5. (a) *Proof.* Suppose H char K and $K \trianglelefteq G$. Since $K \trianglelefteq G$, every inner automorphism of G restricted to K is an automorphism of K by Proposition 4.13. Since H char K, every automorphism of K maps H to itself. In particular, the inner automorphism of G maps H to itself, that is, $gHg^{-1} = H$ for all $g \in G$. So H is normal. □
   (b) *Proof.* Let $\varphi \in \mathrm{Aut}(G)$. Since K char G, $\varphi(K) = K$. So $\varphi \in \mathrm{Aut}(K)$, and since H char K, $\varphi(H) = H$. Thus H char K. □

6. (a) Since $12 = 2^2 \cdot 3$, the Sylow 2-subgroups are the subgroups of order 4. The elements of order 2 in D_{12} are $\{1, r^3, s, sr, sr^2, sr^3, sr^4, sr^5\}$. So the Sylow 2-subgroups are $\langle s, r^3 \rangle$, $\langle r^3, sr \rangle$, $\langle r^3, sr^2 \rangle$ (these comprise all non-identity elements of order 2 and are conjugates).

   Similarily, the Sylow 3-subgroups are the subgroups of order 3. The only elements of orders 1 or 3 are $\{1, r^2, r^4\}$, so this is the sole Sylow 3-subgroup of D_{12}.
   (b) Since $|S_4| = 2^3 \cdot 3$, the Sylow 2-subgroups then are the subgroups of order 8. Let G be a subgroup of S_4 isomorphic to D_8 (by Cayley's Theorem). Then the conjugations of G are the Sylow 2-subgroups: namely the symmetries of a square with vertices labelled $\{1, 2, 3, 4\}$, $\{1, 2, 4, 3\}$, and $\{1, 3, 2, 4\}$. That is, $\langle (1234), (12)(34) \rangle$, $\langle (1243), (12)(43) \rangle$, and $\langle (1324)(13)(24) \rangle$. By Sylow's Theorem, $n_2 \equiv 1 \pmod 2$ and $n_2 \mid 3$, so these are all of them.

   The Sylow 3-subgroups are the subgroups of order 3, which are generated by the 3-cycles in S_4: $\langle (123) \rangle, \langle (134) \rangle, \langle (234) \rangle, \langle (124) \rangle$.

7. (a) $105 = 3 \cdot 7 \cdot 5$. By Sylow's Theorem, $n_7 \equiv 1 \pmod 7$ so $n_7 = 1, 8, 16, \ldots$ and $n_7 \mid 15$, so $n_7 = 1$. Thus there is one Sylow 7-subgroup, so it is normal, by Corollary 20.

   (b) $351 = 3^3 \cdot 13$. By Sylow's Theorem, $n_3 = 1 \pmod 3$ and $n_3 \mid 13$. So $n_3 = 13$ or 1. If $n_3 = 1$ we are done so suppose $n_3 = 13$. Since each Sylow 3-subgroup has $3^3 = 27$ elements, there are $13 \cdot 26 = 338$ distinct non-identity elements with orders that divide 27. This leaves $351 - 338 = 13$ distinct elements. A Sylow 13-subgroup must have order $13^1 = 13$ since the prime factor decomposition of 351 has only 1 13 term, so the 13 distinct elements form the unique Sylow 13-subgroup. So it is normal.

8. *Proof.* Let G be a simple group of order $168 = 7 * 3 * 2^3$. There are $n_7 \equiv 1 \pmod 7$ Sylow 7-subgroups with $n_7 \mid 168$. Since $168/7 = 24$, $n_7 \leq 24$. Since G is simple, the Sylow 7-subgroups are not normal, so $n_7 > 1$. Thus $n_7 = 8$, so there are $6 \cdot 8 = 48$ elements of order 7 (note that the order of each element except the identity in each Sylow 7-subgroup is 7, since 7 is prime). □

9. $n_5 = 6$ since $n_5 \pmod 5$ and $n_5 \mid 60$ and $1 < n_5 \leq 60/5 = 12$, since A_5 is simple. Similarily, n_3 is 10 or 4. But since there are $(5)(4)(3)/3 = 30$ 3-cycles, and each 3-cycle is in a Sylow 3-subgroup, we must have $n_3 = 10$. We've used $(3-1)(10) + (4)(6) = 44$ non-identity elements, leaving 15 non-identity elements left. Finally, $n_2 = 3, 5, 15$ by the same reasoning as above. Since A_5 consists only of 3-cycles, 5-cycles, and the product of 2 2-cycles, the 15 non-identity elements left are products of 2 2-cycles. We can generate $15/3 = 5$ groups of order $2^2 = 4$ with these non-identity elements, so $n_2 = 5$.

10. First a lemma (Exercise 42 in Chapter 3):

    Lemma 1. *Let $H, K \trianglelefteq G$ with $H \cap K = 1$. Then $xy = yx$ for all $x \in H$ and $y \in K$.*

    Proof. Let $x \in H, y \in K$. Then $x^{-1}y^{-1}xy = x^{-1}h \in H$ where $h = y^{-1}xy \in H$ by normality of H. Similarily, $x^{-1}y^{-1}xy = ky \in K$ where $k = x^{-1}y^{-1}x \in K$ by normality of K. So $x^{-1}y^{-1}xy \in H \cap K = 1$, so $xy = yx$. □

    Now we prove the theorem.

    Proof. Let H be a proper, non-trivial normal subgroup of S_n. Since $A_n \trianglelefteq S_n$, we have $H \cap A_n \trianglelefteq S_n$. Since $H \cap A_n \leq A_n$, $H \cap A_n \trianglelefteq A_n$. But, A_n is simple, so either $H \cap A_n = 1$ or $H \cap A_n = A_n$. If $H \cap A_n = A_n$ then, $H \leq A_n \leq S_n$, and because $[S_n : A_n] = [S_n : H][H : A_n] = 2$, one of the indices is 1, so either $H = A_n$ or $H = S_n$, and we are done. So suppose instead $H \cap A_n = 1$. By Lemma 1, $H \subseteq Z(S_n) = 1$ by Question 3. So $H = 1$. □