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HW 5

Math 3120
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1. *Proof.* Let $N \trianglelefteq H$ be a proper, non-trivial normal subgroup of H. Then $N \cap G_i \trianglelefteq G_i$ for all i, so $N \cap G_i = G_i$ or $N \cap G_i = 1$, since G_i is simple. If $N \cap G_i = G_i$ for all i, then $G_i \leq N$, so $N = H$; a contradiction. Similarily, if $N \cap G_i = 1$ for all i, then $H = 1$; a contradiction. So suppose let j, k such that $N \cap G_j = G_j \neq 1$ and $N \cap G_k = 1$. If $j < k$, then $G_j \leq G_k$, so $N \cap G_j \leq N \cap G_k = 1$, contradicting that $G_j \neq 1$. If instead $k < j$, then $G_k \leq G_j$, but $N \cap G_k = 1$ so we can't have $N \cap G_j = G_j$, since that would imply $G_j \leq N$. Thus N can't exist, so H is simple. □

2. (a) Consider $\langle (i,1) \rangle = \{(i,1), (-1,2), (-i,3), (-1,0)\} \leq Q_8 \times Z_4$. Then $(j,0)(i,1)(j,0)^{-1} = (-i,0) \notin \langle (i,1) \rangle$, so it is not normal.

   (b) *Proof.* Let $N \leq G$, $(x,y) \in N, (a,b) \in G$. Then $(a,b)(x,y)(a,b)^{-1} = (axa^{-1}, byb^{-1})$. Since $b \in E_{2^n}$, $byb^{-1} = b$. So, it remains to show that $(axa^{-1}, b) \in N$. Since $a, x \in Q_8$, $axa^{-1} = \pm x$. If $axa^{-1} = x$ we are done. If $axa^{-1} = -x$, note that $(-x,y)^{-1} = (x,y) \in N$ (since $y \in E_{2^n}$), so $(-x,y) \in N$. □

3. (a) We have $\coprod G_i \subseteq \prod G_i$ and $(1, \dots) \in \coprod G_i$ so $\coprod G_i$ is non-empty. Let $x = (x_1, \dots) \in \coprod G_i$ where $x_k = x_{k+1} = \cdots = 1$ for some k and $y = (y_1, \dots) \in \prod G_i$. Then

$$\begin{aligned} yxy^{-1} &= (y_1x_1y_1^{-1}, \dots, y_kx_ky_k^{-1} \dots) \\ &= (y_1x_1y^{-1}, \dots, y_k1y_k^{-1}, \dots) \\ &= (y_1x_1y^{-1}, \dots, 1, \dots) \end{aligned}$$

   so $yxy^{-1} \in \coprod G_i$, and by the subgroup criterion, $\coprod G_i \leq \prod G_i$.

   (b) *Proof.* Let $x \in T(\prod G)$. Then $|x| = n < \infty$. If $x = (x_1, x_2, \dots)$ had infinitely many non-identity components, then its order would be infinite, since one could always choose a prime $p > n$ such that $x_i \in Z_p$ such that $x_i^n \neq 1$. So all but finitely many components of x are the identity, so $x \in \coprod G$. Conversely, if $x \in \coprod G$, then there are finitely many non-identity components each in some cyclic group $Z_{p_1}, \dots, Z_{p_n}$. Then $|x| \leq p_1 \dots p_n < \infty$, so $x \in T(\prod G)$. □

4. (a) *Proof.* Let $|G| = n = p_1^{\alpha_1} \dots p_n^{\alpha_n}$. Then by Sylow's Theorem, G has Sylow p_i-subgroups P_i of order $p_i^{\alpha_i}$. Since n_{p_i} must divide $p_1^{\alpha_1} \dots p_n^{\alpha_n} / p_i^{\alpha_i}$, $n_{p_i} = 1$, so each P_i is the unique Sylow p_i-subgroup and thus normal.

   We proceed by induction on n. Suppose the statement holds for finite abelian groups of order less than n.We have that the subgroup $H \leq G$ generated by $P_2, \dots, P_n$ is the product of its Sylow subgroups $P_2, \dots, P_n$ and is normal since G is abelian. By Lagrange's $H \cap P_1 = 1$ is a direct product, so by the Recognition Theorem $G \cong P_1 \times H \cong P_1 \times \dots P_n$. □

   (b) *Proof.* Note that since H, K char G, if $\phi \in \text{Aut}(G)$ then $\phi\mid_H \in \text{Aut}(H)$ and $\phi\mid_K \in \text{Aut}(K)$. So define $\varphi : \text{Aut}(G) \to \text{Aut}(H) \times \text{Aut}(K)$ by $\varphi(\sigma) = (\sigma|_H, \sigma|_K)$. Then φ is a homomorphism since $\varphi(\sigma\tau) = (\sigma\tau|_H, \sigma\tau|_K) = (\sigma|_H\tau|_H, \sigma|_K\tau|_K) = (\sigma|_H, \sigma|_K)(\tau|_H, \tau|_K) = \varphi(\sigma)\varphi(\tau)$.

   The kernel of φ is the set of automorphisms that fix H and K, that is σ such that $\sigma|_H = 1$ and $\sigma|_K = 1$. Since $G = H \times K$, it must be that $\sigma = 1$. Thus φ is injective.

   To show that φ is surjective, note that given $h \in \text{Aut}(H)$ and $k \in \text{Aut}(K)$, we can define $\sigma \in \text{Aut}(G)$ by $\sigma = h \cup k$ since $H \cap K = 1$. Thus φ is surjective, and thus bijective, so $\text{Aut}(G) \cong \text{Aut}(H) \times \text{Aut}(K)$. □

   (c) *Proof.* By induction on the number of Sylow subgroups of G. The base case of $n = 1$ is trivial. Suppose G is finite abelian so that it has $n > 1$ distinct Sylow subgroups $P_1, \dots, P_n$

and that the statement holds for finite abelian groups of order less than n. Consider G/P_1. By the induction hypothesis, $\mathrm{Aut}(G/P_1) \cong \mathrm{Aut}(P_2)\times\cdots\times\mathrm{Aut}(P_n)$. As in (b), $G/P_i \cap P_i = 1$ and are characteristic, so by (b) $\mathrm{Aut}(G) \cong \mathrm{Aut}(P_1) \times \cdots \times \mathrm{Aut}(P_n)$. □

5. *Proof.* Let G be non-abelian. Recall that if $G/Z(G)$ is cyclic, then G is abelian, so $Z(G)$ must have order p. Thus $|G/Z(G)| = p^2$, so $G/Z(G)$ is abelian by Corollary 4.9. Since G' is the smallest normal subgroup of G with abelian quotient, $G' \leq Z(G)$. Since G is non-abelian, $1 < G' \leq Z(G) < G$, but $|G'| = |Z(G)| = p$, so $G' = Z(G)$. □

6. (We use the notation that $x^g = gxg^{-1}$).
   (a) *Proof.* Let $g \in G$ and $x, y \in K$. Then $x = a^g$ and $y = b^g$ for some $a, b \in K$, so $(x^{-1}y^{-1}xy)^g = (a^{-g}b^{-g}(a^{-1})^{-g}(b^{-1})^{-g})^g = aba^{-1}b^{-1} \in K'$. □
   (b) *Proof.* Let $\varphi : G \to \mathrm{Aut}(K)$ be the permutation representation of G associated with the action by conjugation. Then it is a homomorphism, and since $\mathrm{Aut}(K)$ is Abelian, by Proposition 5.7(5), $G' \leq \ker\varphi = C_G(K)$. □

7. (a) *Proof.* By Theorem 5.10(5),

$$\begin{aligned} k \in C_K(H) &\iff h^k = k, \forall h \in H \\ &\iff \varphi(k)(h) = h, \forall h \in H \\ &\iff \varphi(k) = 1 \\ &\iff k \in \ker\varphi \end{aligned}$$

□

   (b) *Proof.* Let $h \in H$ and $k \in K$. Then

$$(1, k)^{(h,1)} = (h, k)(h, 1)^{-1} = (h, k)(h^{-1}, 1) = (hk \cdot h^{-1}, k)$$

If also $h \in N_H(K)$, then $(1, k)^{(h,1)} = (1, k') = (hk \cdot h^{-1}, k)$. So $k' = k$, so $(1, k)^{(h,1)} = (1, k)$, so $h \in C_H(K)$. Clearly $C_H(K) \leq N_H(K)$, so $C_H(K) = N_H(K)$. □

8. (a) *Proof.* We need to show that $\mathrm{Aut}(H) \cong S_3$. Note that $Z_2 \times Z_2 \cong V_4$ so $\mathrm{Aut}(Z_2 \times Z_2) \cong \mathrm{Aut}(V_4)$, and the automorphisms of V_4 are the permutations of the 3 non-identity elements. Thus $\mathrm{Aut}(V_4) \cong S_3$. So $|G| = |H||K| = 4 \cdot 6 = 24$, so $G \cong S_4$. □
   (b) *Proof.* Let G act on the left cosets of K by left multiplication. The permutation representation π afforded by this action is a homomorphism $\pi : G \to S_{G/K} \cong S_H \cong S_4$. To show that $\ker\pi = 1$, note that from Exercise 7a, $C_K(H) = \ker(\varphi) = 1$ where $\varphi = 1$ is the homomorphism associated with $G = H \rtimes K$. So, it remains to show that $\ker\pi \leq C_K(H)$.

   Let $g \in \ker\pi$. Then $gxK = xK$ for all $x \in G$. Then $xgx^{-1} \in K$ so $xgx^{-1}g^{-1} \in K$, since $g \in K$ because $g(1)K = (1)K$. If $x \in H$ then since $H \trianglelefteq G$ (by Theorem 5.10), $xgx^{-1}g^{-1} \in H$. But $H \cap K = 1$, again by Theorem 5.10, so $xgx^{-1}g^{-1} = 1$ so $gx = xg$. Thus $g \in C_K(H)$, so $\ker\pi \leq C_K(H)$

   Thus, the kernel of the permutation representation is trivial, so $G \cong S_4$. □