Mathematics 3120B **Assignment 2** 9 February 2024

Please upload your assignment onto Gradescope.ca before 10:00 a.m. on the due date shown above. No late assignments will be accepted. You must do your own work. Write all answers in complete sentences.

1.

(a) Prove that $\mathbb{Q}$ is not finitely generated.
(b) Exhibit a proper subgroup of $\mathbb{Q}$ that is not cyclic.
(c) Prove that $\mathbb{Q}^+$ is generated by the set $\{1/p \,|\, p \text{ is prime}\}$.

2. You may use the lattice of subgroups of Q_8 given in the notes (or in the text on p.69) to help you with the following:

(a) Find the centralizer and normalizer of each subgroup of Q_8.
(b) Which subgroups of Q_8 are normal? For each normal subgroup N of Q_8, find the isomorphism type of its quotient.

3. A nontrivial Abelian group A is called *divisible* if for each $a \in A$ and nonzero integer n there is an $x \in A$ such that $x^n = a$.

(a) Prove that the quotient of a divisible group A by any proper subgroup B is divisible.
(b) Prove or disprove each of the following groups are divisible: a finite Abelian group G, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$.

4. Let A be a group, and let D be the *diagonal* subgroup $\{(a, a) \,|\, a \in A\}$ of $A \times A$.

(a) Suppose that A is Abelian. Prove that D is a normal subgroup of $A \times A$.
(b) Suppose that A is the non-Abelian group S_3. Prove that D is not normal in $A \times A$.

5. Let G be a group.

(a) Prove that, if $G/Z(G)$ is cyclic, then G is Abelian.
(b) Prove that, if $|G| = pq$ for some (possibly equal) primes p and q, then either G is Abelian or $Z(G) = 1$.

6. Let G be a group, let $H \leq G$, and let $x, y \in G$. Prove that, if $xH = Hy$, then $xH = Hx$ and $x \in N_G(H)$.

7. Let G be a finite group, let $H \leq G$, and let N be a normal subgroup of G. Prove that, if $(|H|, |G : N|) = 1$, then $H \leq N$.

8. Prove that, if M is a normal subgroup of G such that $|G : M| = p$ is prime, then, for all $H \leq G$, either:

(i) $H \leq M$ or
(ii) $G = MH$ and $|H : H \cap M| = p$.

9. Let M and N be normal subgroups of G such that $G = MN$. Prove that

$$G/(M \cap N) \cong (G/M) \times (G/N).$$