Mathematics 3120B **Assignment 4** 22 March 2024

*Please upload your assignment onto Gradescope.ca before 10:00AM on the due date shown above. You must do your own work. No late assignments will be accepted. Justify all of your answers. Write all answers in complete sentences.*

**1.** Find the conjugacy classes of the following groups:

(a) $Q_8$

(b) $A_4$

**2.**

(a) If $|G : Z(G)| = n$, prove that the conjugacy class of $x \in G$ has at most $n$ elements.

(b) Find all finite groups with exactly two conjugacy classes. Justify your answer.

**3.** Prove that $Z(S_n) = 1$, for all $n \geq 3$.

**4.** Let $G$ be a group of order 203. Prove that, if $G$ has a normal subgroup $H$ of order 7, then $G$ is Abelian.

**5.** Suppose that $H \leq K \leq G$.

(a) Prove that, if $H$ is characteristic in $K$ and $K$ is normal in $G$, then $H$ is normal in $G$.

(b) Prove that being characteristic is transitive; in other words, if $H$ is characteristic in $K$ and $K$ is characteristic in $G$, then $H$ is characteristic in $G$.

**6.** List the distinct Sylow 2-subgroups and Sylow 3-subgroups of the following groups.

(a) $D_{12}$

(b) $S_4$

**7.** Prove that every group $G$ with the following orders has a normal Sylow $p$-group, for some prime $p$ dividing its order.

(a) 105

(b) 351

**8.** How many elements of order 7 must there be in a simple group of order 168?

**9.** For $p = 2$, 3 and 5, find $n_p(A_5)$.

**10.** Let $n \geq 5$. Prove that, if $N$ is a normal subgroup of $S_n$, then $N$ is 1, $A_n$ or $S_n$.