Mathematics 3120B **Assignment 5** 5 April 2024

Please upload your assignment onto Gradescope.ca before 10:00AM on the due date shown above. You must do your own work. No late assignments will be accepted. Justify all of your answers. Write all answers in complete sentences.

1. Prove that, if $G_1 \le G_2 \le \cdots$ is a chain of simple subgroups of G, then $H = \cup_{i=1}^{\infty} G_i$ is a simple subgroup of G, too.

2.

(a) Exhibit a nonnormal subgroup of $G = Q_8 \times Z_4$. Note: every subgroup of each factor is normal.

(b) Show that all subgroups of $G = Q_8 \times E_{2^n}$ are normal. Hint: you don't need to find its subgroups to show that they must be closed under conjugation.

3.

(a) Let I be a nonempty index set, and let G_i be a group for each $i \in I$. Prove that $\coprod_{i \in I} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$.

(b) Let $G_i = Z_{p_i}$, where p_i is the i^{th} prime integer. Prove that $\coprod_{i \in \mathbb{Z}^+} G_i$ is the torsion subgroup of $\prod_{i \in \mathbb{Z}^+} G_i$. Recall: if a group G is Abelian, then the subset of all elements of finite order forms the *torsion subgroup*, $T(G)$, of G.

4.

(a) Prove that a finite Abelian group G is the direct product of its Sylow subgroups.

(b) Prove that if $G = HK$, where H and K are characteristic subgroups of G with $H \cap K = 1$, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(H) \times \mathrm{Aut}(K)$.

(c) Deduce that, if G is a finite Abelian group, then $\mathrm{Aut}(G)$ is isomorphic to the direct product of the automorphisms groups of its Sylow subgroups.

5. Prove that if p is a prime and G is a group of order p^3 then either G is Abelian or $G' = Z(G)$.

6. Let K be a normal subgroup of G.

(a) Prove that K' is normal in G, too.

(b) Prove that, if K is cyclic, then $G' \le C_G(K)$. Hint: $\mathrm{Aut}(K)$ is Abelian since K is cyclic.

7. Let H and K be groups, let $\varphi : K \to \mathrm{Aut}(H)$ be a homomorphism, and identify H and K as subgroups of $G = H \rtimes_\varphi K$ in the usual way.

(a) Prove that $C_K(H) = \ker(\varphi)$.

(b) Prove that $C_H(K) = N_H(K)$.

8. Let $G = \mathrm{Hol}(Z_2 \times Z_2)$.

(a) Prove that $G = H \rtimes K$, where $H = Z_2 \times Z_2$ and $K \cong S_3$. Deduce that $|G| = 24$.

(b) Prove that $G \cong S_4$. Hint: construct a homomorphism $\psi : G \to S_4$ by letting G act on the set of left cosets of K in G. Use the previous exercise to prove this representation is faithful.