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HW 2

1. (a) *Proof.* Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{2N} < \epsilon$. Note that $\int_0^1 f_n(x)dx = \frac{n-1}{2n}$ and $f_m(x) \geq f_n(x)$ if $m > n$. So, for $m, n \geq N$, where $m \geq n$,

$$\begin{aligned}\int_0^1 |f_m(x) - f_n(x)|dx &= \int_0^1 f_m(x)dx - \int_0^1 f_n(x)dx \\ &= \frac{m-n}{2mn} \\ &= \frac{m-n}{2(n+(m-n))n} \\ &\leq \frac{n-m}{2(m-n)n} \\ &= \frac{1}{2n} \\ &\leq \frac{1}{2N} \\ &< \epsilon\end{aligned}$$

So (f_n) is Cauchy in $(C[0,1], d)$. □

To show that (f_n) diverges in $(C[0,1], d)$, we first show that (f_n) converges to the step function

$$g(x) = \begin{cases} 0, & 0 \leq x < 1/2 \\ 1, & 1/2 \leq x \leq 1 \end{cases}$$

in $(\mathbb{R}^{[0,1]}, d)$, where $\mathbb{R}^{[0,1]}$ is the set of functions from $[0,1] \to \mathbb{R}$.

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{2N} < \epsilon$. Since $g(x) \geq f_n(x)$ for all $n \in \mathbb{N}$, $d(f_n, g) = \int_0^1 g(x)dx - \int_0^1 f_n(x)dx. = \frac{1}{2} - \frac{n-1}{2n} = \frac{1}{2n} \leq \frac{1}{2N} < \epsilon$. □

Since $g \notin C[0,1]$ (the step function is discontinuous at $x = 1/2$), by uniqueness of convergence, (f_n) does not converge in $(C[0,1], d)$.

(b) *Proof.* Choose $\epsilon = 1/2$. Let $N \in \mathbb{N}$. Choose $m, n \in \mathbb{N}$ such that $m = 2n > n > N$. Then, for $x \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{m}\right]$,

$$\begin{aligned}|f_m(x) - f_n(x)| &= mx - \frac{m}{2} - \left(nx - \frac{n}{2}\right) \\ &= 2nx - n - nx + \frac{n}{2} \\ &= nx - \frac{n}{2}\end{aligned}$$

so $\left|f_m\left(\frac{1}{2} + \frac{1}{m}\right) - f_n\left(\frac{1}{2} + \frac{1}{m}\right)\right| = \left|f_m\left(\frac{1}{2} + \frac{1}{2n}\right) - f_n\left(\frac{1}{2} + \frac{1}{2n}\right)\right| = \frac{1}{2}$. Thus, $\sup_{x\in[0,1]} |f_m(x) - f_n(x)| \geq \frac{1}{2} = \epsilon$. So (f_n) is not Cauchy. □

2. *Proof.* First we show d_1 is a metric.

   (a) $d_1((x,y),(x',y')) \geq 0$ since $d_X(x,x') \geq 0$ and $d_Y(y,y') \geq 0$.

   (b) $d_1((x,y),(x,y)) = d_X(x,x) + d_Y(y,y) = 0$.

   (c) $d_1((x,y),(x',y')) = d_X(x,x') + d_Y(y,y') = d_X(x',x) + d_Y(y',y) = d_1((x',y'),(x,y))$.

   (d) $d_1((x,y),(x',y')) + d_1((x',y'),(x'',y'')) = d_X(x,x') + d_Y(y,y') + d_X(x',x'') + d_Y(y',y'') \geq d_X(x,x'') + d_Y(y,y'') = d_1((x,y),(x'',y''))$

   Next we show d_∞ is a metric.

   (a) $d_\infty((x,y),(x',y')) \geq 0$ since $d_X(x,x'), d_Y(y,y') \geq 0$.

   (b) $d_\infty((x,y),(x,y)) = \max\{d_X(x,x), d_Y(y,y)\} = 0$.

   (c) $d_\infty((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\} = \max\{d_X(x',x), d_Y(y',y)\} = d_\infty((x',y'),(x,y))$.

   (d) Note that if $d_X(x,x') \geq d_Y(y,y')$ and $d_X(x',x'') \geq d_Y(y',y'')$, then $L = d_\infty((x,y),(x',y')) + d_\infty((x',y'),(x'',y'')) = d_X(x,x') + d_X(x',x'') \geq d_X(x,x'')$. Similarily, if $d_Y(y,y') \geq d_X(x,x')$ and $d_Y(y',y'') \geq d_X(x,x'')$, then $L \geq d_Y(y,y'')$. If $d_X(x,x') \geq d_Y(y,y')$ and $d_Y(y',y'') \geq d_X(x',x'')$, then $L = d_X(x,x') + d_Y(y',y'') \geq d_X(x,x') + d_X(x',x'') \geq d_X(x,x'')$. Finally if $d_Y(y,y') \geq d_X(x,x')$ and $d_X(x',x'') \geq d_Y(y',y'')$, then $L = d_Y(y,y') + d_X(x',x'') \geq d_X(x,x'')$. So, we have $L \geq d_X(x,x''), d_Y(y,y'')$; that is, $L \geq \max\{d_X(x,x''), d_Y(y,y'')\} = d_\infty((x,y),(x'',y''))$.

   □

Proof. ($\rightarrow$) Let $S \subseteq (X \times Y)$ be open with respect to d_1. Let $p = (p_x, p_y) \in S$. Then, there is an $\epsilon > 0$ such that $B_{d_1}(p;\epsilon) \subseteq S$. Let $(x,y) \in B_{d_\infty}(p;\epsilon/2)$. Then $d_X(x,p_x), d_Y(y,p_y) \leq \epsilon/2$, so $d_X(x,p_x) + d_Y(y,p_y) \leq \epsilon$ and $(x,y) \in B_{d_1}(p;\epsilon)$. So $B_{d_\infty}(p;\epsilon/2) \subseteq B_{d_1}(p;\epsilon) \subseteq S$, so S is open with respect to d_∞.

($\leftarrow$) Let S be open with respect to d_∞. Let $p = (p_x, p_y) \in S$. Then, there is an $\epsilon > 0$ such that $B_{d_\infty}(p;\epsilon) \subseteq S$. Let $(x,y) \in B_{d_1}(p;\epsilon)$. Then, $d_X(x,p_x) + d_Y(y,p_y) \leq \epsilon$, so $d_X(x,p_x), d_Y(y,p_y) \leq \epsilon$. So $\max\{d_X(x,p_x), d_Y(y,p_y)\} \leq \epsilon$, and $(x,y) \in B_{d_\infty}(p;\epsilon)$. So $B_{d_1}p;\epsilon \subseteq B_{d_\infty}(p;\epsilon) \subseteq S$, so S is open with respect to d_1. □