David Tran
251169871

HW 3

Math 3122
October 24, 2023

1. First we show that $B(0,2)$ can be covered by finitely many open balls.

   Proof. Define 3 open balls in $\mathbb{R}$ on the standard metric as $B_1(0,1)$, $B_2(-1,1)$, $B_3(1,1)$. Note that $B_1 = (-1,1)$, $B_2 = (-2,0)$, and $B_3 = (0,2)$. Let $x \in B(0,2)$. Then $x \in (-2,2)$ so $x \in \bigcup_{k=1}^{3} B_k = (-2,2)$. That is, B is covered by finitely many open balls. □

   Next we show that this doesn't hold in ℓ_1.

   Proof. Consider the sequences $X = \{(x_n)_k \mid k \in \mathbb{N}\}$ defined such that $x_n = 3/2$ when $n = k$ and $x_n = 0$ otherwise. $X \subseteq \ell_1$ since only $x_k = 3/2$ is non-zero, so every series converges to $3/2$. Also, $X \subseteq B(0;2)$, since for all $x \in X$, $d(x,0) = 3/2 < 2$ for all $k \in \mathbb{N}$. So, any finite covering of $B(0;2)$ must at least cover X. Note that any two distinct sequences $x, y \in X$ cannot be contained in a single open ball of radius 1. To see this, suppose for contradiction that $x, y \in B(z;1)$ for some sequence z. Then $d(x,z), d(y,z) < 1$, so $d(x,z) + d(z,y) < 2$. But, $d(x,y) = 3$, so $d(x,y) > d(x,z) + d(z,y)$, violating the triangle inequality; a contradiction. Thus, every open ball of radius 1 can only contain one sequence in X. So, since $|X| = |\mathbb{N}|$, X requires infinitely many open balls of radius 1 to be covered. So $B(0;1)$ requires infinitely many such open balls to be covered. □

2. *Proof.* Let $f : X \to X$ be a contraction on a metric space with metric d. Then there is an $\alpha \in \mathbb{R}$, $0 < \alpha < 1$ such that $d(f(x), f(y)) = \alpha d(x,y)$. Let $\epsilon > 0$. Fix $\delta = \epsilon/\alpha$. Then for all $x, y \in X$, if $d(x,y) < \delta$, then $d(f(x), f(y)) = \alpha d(x,y) < \alpha \cdot \frac{\epsilon}{\alpha} = \epsilon$. So f is uniformly continuous. □