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HW 4

Math 3122
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1. *Proof.* Let A and B be connected and suppose for the sake of contradiction that $A \cup B$ is disconnected. Then there exists open, disjoint, non-empty subsets $U, V \subseteq A \cup B$ such that $A \cup B = U \cup V$. Choose $x \in A \cap B$. Then $x \in U$ or $x \in V$. Without loss of generality, suppose $x \in U$ and choose $y \in V$. Then $y \in A$ or $y \in B$. Without loss of generality, suppose $y \in A$. Since $x \in A \cap B$ and $x \in U$, then $x \in A \cap U$. Also, $y \in A \cap V$. We have that $A \subseteq A \cup B = U \cup V$. So neither U nor V are empty. Thus, since $A \subseteq A \cup B = U \cup V$, U and V are open, disjoint, non-empty subsets that cover A, contradicting the fact that A is connected. □

2. *Proof.* $(\rightarrow)$ Suppose X is connected. Then the only subsets of X that are clopen are $\emptyset$ and X. Since f is a homeomorphism, it is continuous, so it preserves openess and closedness. Thus the only subsets of $f(X) = Y$ that are clopen are $f(\emptyset) = \emptyset$ and Y. So Y is connected.

   $(\leftarrow)$ Since f is a homeomorphism, it is bijective, so it has a continuous inverse. Then the same proof as above applies for $Y \rightarrow X$. □