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# HW 5

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1. *Proof.* Let $U$ be an open cover of $X$. Since $U$ is open, for every $x \in U$, there is an open ball $B_x$ of radius $\epsilon(x)$ around $x$ contained in $U$. Consider the union of all such balls $V = \bigcup_{x \in U} B_x$. Since every ball is contained in $U$, $V \subseteq U$. Also, since every $x \in U$ is contained in some ball $B_x$, $U \subseteq V$. Thus, $U = V$, so $V$ is a cover of $X$ by open balls, so it has a finite subcover. So $U$ has a finite subcover, so $X$ is compact. □

2. *Proof.* Suppose $X$ is not compact. Let $(x_n)$, $n \in \mathbb{N}$ be an arbitrary sequence with no accumulation point in $X$. Define $f(x) = \inf_n\{d(x, x_n) + 1/n\}$. First we show that $f$ is continuous. Let $x, y \in X$ with $d(x, y) > 0$. We have that for $n \in \mathbb{N}$,

$$f(x) \leq d(x, x_n) + 1/n \leq d(x, y) + d(y, x_n) + 2/n$$

so $f(x) \leq d(x, y) + f(y) + 1/n$. Similarily, $f(y) \leq d(x, y) + f(x) + 1/n$. So,

$$|f(x) - f(y)| \leq d(x, y) + 1/n$$

Since $1/n$ and $d(x, y)$ can be made arbitrarily small, $f$ is continuous.

Next we show that $f(X)$ is non-compact. Note that $0 \notin f(X)$ since if $x \notin (x_n)$, then $f(x) > 0$ since $x$ is not an accumulation point, so $\lim_{n\to\infty} d(x, x_n) \neq 0$. If instead $x \in (x_n)$, then

$$f(x) = \inf\{1/n : x_n = x\} = \min\{1/n : x_n\} > 0$$

since there are finitely many $x_n = x$. We know there are only finitely many such $x_n$ since if there weren't, $(x_n)$ would have an accumulation point $x$. Finally, note that $1/n \in f(X)$ for arbitrarily large $n$ since $f(y) = 1/n$, where $n = \max\{n : y = x_n\}$, which exists because there are only finitely many such $x_n = y$ by $y$ not being an accumulation point. So, we can form a sequence $(f(y_k)) \subseteq f(X)$ converging to 0. But, $0 \notin f(X)$. So $f(X)$ is not closed, and by Heine-Borel, is not compact.

□