III. REAL NUMBERS (cf. Ch. 3)

1. ORDERED FIELDS

Def. A *field* is a triple $(\mathbb{F}, +, \cdot)$ where $\mathbb{F}$ is a nonempty (!) set, "$+$": $\mathbb{F}\times\mathbb{F} \to \mathbb{F}$ and "$\cdot$": $\mathbb{F}\times\mathbb{F}\to\mathbb{F}$ are functions, called *addition* and *multiplication*, satisfying the following axioms:

(A1) $\forall x,y \in \mathbb{F}$, $x+y\in\mathbb{F}$

(A2) $\forall x,y\in\mathbb{F}$, $x+y=y+x$ /commutativity/

(A3) $\forall x,y,z\in\mathbb{F}$, $x+(y+z)=(x+y)+z$ /associativity/

(A4) there exists an element $0\in\mathbb{F}$ st. $\forall x\in\mathbb{F}$, $x+0=x$ /additive identity/

(A5) $\forall x\in\mathbb{F}$ $\exists y\in\mathbb{F}$ st. $x+y=0$. We write $y=-x$. /additive inverse/

(M1) $\forall x,y\in\mathbb{F}$, $x\cdot y\in\mathbb{F}$

(M2) $\forall x,y\in\mathbb{F}$, $x\cdot y=y\cdot x$

(M3) $\forall x,y,z\in\mathbb{F}$, $x\cdot(y\cdot z)=(x\cdot y)\cdot z$

(M4) there exists an element $1\in\mathbb{F}\setminus\{0\}$ st. $\forall x\in\mathbb{F}$, $x\cdot 1=x$ /multiplicative identity/

(M5) $\forall x\in\mathbb{F}\setminus\{0\}$ $\exists y\in\mathbb{F}$ st. $x\cdot y=1$. We write $y=x^{-1}$ (or $\frac{1}{x}$)

(DL) $\forall x,y,z\in\mathbb{F}$, $x\cdot(y+z)=x\cdot y+x\cdot z$. /distributive law/

(cf. 3.4) **Thm.** Let $(\mathbb{F},+,\cdot)$ be a field. Then,

(i) The additive and multiplicative identities are unique.

(ii) $\forall x\in\mathbb{F}$, $-x$ is unique.

(iii) $\forall x\in\mathbb{F}\setminus\{0\}$, x^{-1} is unique.

(iv) $\forall x,y,z\in\mathbb{F}$, $(x+z=y+z)\Rightarrow(x=y)$ /cancelation law/

(v) $\forall x\in\mathbb{F}$, $x\cdot 0=0$

(vi) $\forall x\in\mathbb{F}$, $(-1)\cdot x=-x$

(vii) $\forall x,y\in\mathbb{F}$, $x\cdot y=0\Rightarrow(x=0 \vee y=0)$

(viii) $\forall x,y\in\mathbb{F}$ $\exists!\, z\in\mathbb{F}$ st. $x=y+z$ /subtraction/

(ix) $\forall x\in\mathbb{F}$ $\forall y\in\mathbb{F}\setminus\{0\}$ $\exists!\, z\in\mathbb{F}$ st. $x=y\cdot z$ /division/

Pf. (i) Suppose $0_1, 0_2$ both satisfy (A4). Then, $0_1 \overset{(A4)\text{ for }0_2}{=} 0_1+0_2 \overset{(A2)}{=} 0_2+0_1 \overset{(A4)\text{ for }0_1}{=} 0_2$. ✓

Similarly, if $1_1, 1_2$ both satisfy (M4), then $1_1 \underset{(M4)\text{ for }1_2}{=} 1_1\cdot 1_2 \underset{(M2)}{=} 1_2\cdot 1_1 \underset{(M4)\text{ for }1_2}{=} 1_2$. ✓

(ii) Given $x \in \mathbb{F}$, suppose $x+y=0 \wedge x+z=0$. Then, $y=y+0=y+(x+z)=(x+y)+z=0+z=z$. ✓

(iii) Given $x \in \mathbb{F}\setminus\{0\}$, suppose $x\cdot y=1 \wedge x\cdot z=1$. Then, $y=y\cdot 1=y\cdot(x\cdot z)=(x\cdot y)\cdot z=1\cdot z=z\cdot 1=z$. ✓

(iv) Let x,y,z be s.t. $x+z=y+z$. Then, $x=x+0=x+(z+(-z))=(x+z)+(-z)=y+(z+(-z))$ $=y+0=y$. ✓

(v) Given $x\in\mathbb{F}$, we have $x\cdot 0=x\cdot(0+0)=x\cdot 0+x\cdot 0$. On the other hand, $x\cdot 0=x\cdot 0+0$, hence $0+x\cdot 0=x\cdot 0+x\cdot 0$, and thus $0=x\cdot 0$, by (iv). ✓

(vi) Given $x\in\mathbb{F}$, $(-1)\cdot x+x=((-1)+1)\cdot x=0\cdot x=0$, by (v). By (ii) then, $(-1)\cdot x=-x$. ✓

(vii) Suppose $x,y\in\mathbb{F}\setminus\{0\}$ and $x\cdot y=0$. Then, both x and y have multiplicative inverses and hence $\underset{\uparrow \text{ by (M4)}, 1\neq 0}{1} = 1\cdot 1 = x^{-1}\cdot x\cdot y\cdot y^{-1}=x^{-1}\cdot 0\cdot y^{-1}=x^{-1}\cdot 0=0$, by (v). This contradicts (M4). ✓

(viii) Given $x,y\in\mathbb{F}$, define $z=x+(-y)$. Then, $x=x+0=x+(y+(-y))=y+(x+(-y))=y+z$. If, for some other $w\in\mathbb{F}$, $x=y+w$, then $y+w=y+z \overset{(iv)}{\Longrightarrow} w=z$. ✓

(ix) Given $x\in\mathbb{F}$ and $y\in\mathbb{F}\setminus\{0\}$, define $z=x\cdot(y^{-1})$. Then $x=x\cdot 1=x\cdot(y\cdot y^{-1})=y\cdot(x\cdot y^{-1})=y\cdot z$. If also $x=y\cdot w$, then $w=1\cdot w=y^{-1}\cdot yw=y^{-1}x=y^{-1}\cdot y\cdot z=1\cdot z=z$. ∎

<u>Def.</u> The <u>characteristic</u> of a field $\mathbb{F}$ is defined as

$$\text{char}(\mathbb{F})=\begin{cases} p, & \text{if } p=\min\{k\in\mathbb{N} \mid \overbrace{1+1+\ldots+1}^{k \text{ times}}=0\} \\ 0, & \text{if there's no such } p\end{cases}$$

Examples:

1) $\mathbb{Q}=\{p/q : p\in\mathbb{Z}, q\in\mathbb{N}_+\}$ = rational numbers — the smallest field of characteristic 0.

2) $\mathbb{R}$, $\mathbb{C}$ — other fields of characteristic 0

3) If $p\in\mathbb{N}$ is a prime, define $\mathbb{Z}_p:=\{\bar{0},\bar{1},\bar{2},\ldots,\overline{p-1}\}$ with addition and multiplication induced from $\mathbb{Z}$ modulo p (i.e., $\bar{m}+\bar{n}:=\overline{m+n}$, $\bar{m}\cdot\bar{n}:=\overline{mn}$). Then, $\mathbb{Z}_p$ is a field and $\text{char}(\mathbb{Z}_p)=p$.

<u>Def.</u> An <u>ordered field</u> is a field equipped with a linear order relation compatible with the field addition and multiplication. That is, $(\mathbb{F},+,\cdot)$ is an ordered field, when there is a relation "$<$" on $\mathbb{F}$ satisfying the following

(O1) $\forall x,y\in\mathbb{F}$, $x<y \vee x=y \vee y<x$. /trichotomy/

(O2) $\forall x,y,z\in\mathbb{F}$, $x<y \wedge y<z \Rightarrow x<z$ /transitivity/

(O3) $\forall x,y,z\in\mathbb{F}$, $x<y \Rightarrow x+z<y+z$.

(O4) $\forall x,y,z\in\mathbb{F}$, $x<y \wedge 0<z \Rightarrow x\cdot z<y\cdot z$

<u>Notation</u>: We write $x\leq y$, when $x<y \vee x=y$.

Examples: $\mathbb{Q}$, $\mathbb{R}$ with usual $<$. But, $\mathbb{C}$ or $\mathbb{Z}_p$ are <u>not</u> ordered fields!

Def. Let $(\mathbb{F},<)$ be an ordered field. We say that an element $a\in\mathbb{F}$ is positive when $0<a$, and negative when $a<0$. (Also, nonnegative when $0\leq a$.)

(cf. 3.5) **Thm.** Let $(\mathbb{F},<)$ be an ordered field, $x,y,z,w\in\mathbb{F}$. Then,

(i) $(x<y \wedge z<w) \Rightarrow x+z<y+w$.

(ii) $x<y \Rightarrow -y<-x$.

(iii) $(x<y \wedge z<0) \Rightarrow x\cdot z > y\cdot z$

(iv) $0<1$.

(v) $x>0 \Rightarrow 1/x>0$.

(vi) $0<x<y \Rightarrow 1/x > 1/y$.

Pf. (i) Suppose $x<y \wedge z<w$. Then, $x+z \overset{(O3)}{<} y+z = z+y \overset{(O3)}{<} w+y = y+w$. ✓

(ii) Suppose $x<y$. Then, $-y = -y+0 = -y+(x+(-x)) = x+(-y+(-x)) < (y+(-y))+(-x) = 0+(-x) = -x$. ✓

(iii) By (ii), if $z<0$, then $-z>-0$. But $-0=0$, so $0<-z$. Then,
$x<y \overset{(O4)}{\Rightarrow} x\cdot(-z)<y\cdot(-z) \Rightarrow -xz<-yz \overset{(ii)}{\Rightarrow} -(-yz)<-(-xz) \Rightarrow yz<xz$. ✓ by uniqueness of additive inverse

(iv) By definition $0\neq 1$. Suppose then that $1<0$.
Then, $1=1\cdot 1 \overset{(iii)}{>} 1\cdot 0 = 0$ which contradicts $1<0$. Thus, by (O1), $0<1$. ✓

(v) Suppose $x>0$. Then, $\neg(\frac{1}{x}=0)$, for else $1=x\cdot\frac{1}{x}=x\cdot 0=0$; a contradiction.
By (O1) thus $\frac{1}{x}<0$ or $\frac{1}{x}>0$. Suppose $\frac{1}{x}<0$. Then, by (iii),
$x>0 \Rightarrow 1=x\cdot\frac{1}{x}<0\cdot\frac{1}{x}=0$, which contradicts (iv). Thus, $\frac{1}{x}>0$. ✓

(vi) Suppose $0<x<y$. Then, $\neg(\frac{1}{x}=\frac{1}{y})$, for else $1=x\cdot\frac{1}{x}=x\cdot\frac{1}{y}<y\cdot\frac{1}{y}=1$; a contradiction
Thus, by (O1), $\frac{1}{x}<\frac{1}{y}$ or $\frac{1}{y}<\frac{1}{x}$ by (O4) & (v)
Suppose $\frac{1}{x}<\frac{1}{y}$. Then, $1=x\cdot\frac{1}{x}<x\cdot\frac{1}{y}<y\cdot\frac{1}{y}=1$, a contradiction. Thus, $\frac{1}{y}<\frac{1}{x}$. ∎

Thm. Let $(\mathbb{F},<)$ be an ordered field. Then, there is an injection $\mathbb{N}\xrightarrow{\varphi}\mathbb{F}$, such that elements of $\varphi(\mathbb{N}_+)$ are positive, and $\mathrm{char}(\mathbb{F})=0$.

Pf. Define a function $\varphi:\mathbb{N}\to\mathbb{F}$ recursively by $\varphi(0_{\mathbb{N}}):=0_{\mathbb{F}}$, $\varphi(n+1_{\mathbb{N}}):=\varphi(n_{\mathbb{N}})+1_{\mathbb{F}}$ for all $n\in\mathbb{N}_+$. Then, by above thm., $\forall n\in\mathbb{N}$, $\varphi(n)=0+\varphi(n)<1+\varphi(n)=\varphi(n+1)=0+\varphi(n+1)<1+\varphi(n+1)=\varphi(n+2)<\ldots$ One easily proves by induction that $\varphi(n)<\varphi(n+k)$, $\forall k\in\mathbb{N}_+$
Thus, φ is injective. In particular, there is no $n\in\mathbb{N}_+$ with $\varphi(n)=0$, hence $\mathrm{char}(\mathbb{F})=0$, by definition. ∎

Corollary. Every ordered field $\mathbb{F}$ contains the field of rational numbers $\mathbb{Q}$.

Pf. Let $(\mathbb{F}, <)$ be an ordered field. Then, $\mathbb{N} \subset \mathbb{F}$ and hence, $\forall n \in \mathbb{N}_+,\ \frac{1}{n} \in \mathbb{F}$ — (by (M5))
Similarly, by (A5), $\forall n \in \mathbb{N},\ -n \in \mathbb{F}$. Thus, by (M1), $\forall m, n \in \mathbb{N}_+,\ \frac{m}{n}, -\frac{m}{n} \in \mathbb{F}$. ■

Corollary. Let $(\mathbb{F}, <)$ be an ordered field, $x, y \in \mathbb{F}$.
If $\forall \varepsilon > 0,\ x \leq y + \varepsilon$, then $x \leq y$.

Pf. Suppose $\forall \varepsilon > 0,\ x \leq y + \varepsilon$ and $y < x$. Then $x - y > 0$, and so $\varepsilon := \frac{1}{2} \cdot (x-y) > 0$.
Now, $y + \varepsilon = y + \frac{1}{2} \cdot (x-y) = \frac{1}{2} \cdot 2 \cdot y + \frac{1}{2} \cdot (x-y) = \frac{1}{2} \cdot (2y + x - y) = \frac{1}{2} \cdot ((1+1) \cdot y + x - y) = \frac{1}{2}(y + x + (y-y))$
$= \frac{1}{2}(y+x) < \frac{1}{2} \cdot (x+x) = \frac{1}{2} \cdot 2 \cdot x = x$; a contradiction. ■

Def. Let $(\mathbb{F}, <)$ be an ordered field. Define the <u>absolute value</u> function on $\mathbb{F}$ as

$$|x| := \begin{cases} x, & \text{when } 0 \leq x \\ -x, & \text{when } x < x. \end{cases}$$

Thm. Let $(\mathbb{F}, <)$ be an ordered field, $x, y \in \mathbb{F}$, $a \in \mathbb{F}$, $a \geq 0$. Then,

(i) $|x| \geq 0$
(ii) $|x| \leq a \iff -a \leq x \leq a$
(iii) $|x \cdot y| = |x| \cdot |y|$
(iv) $|x+y| \leq |x| + |y|$ /triangle inequality/

Pf. (i) By definition, and since $x > 0 \Rightarrow -x = x \cdot (-1) < 0$. ✓

(ii) Suppose $|x| \leq a$. If $x \geq 0$, then $x = |x| \leq a$. Also, $a \geq 0 \Rightarrow -a \leq 0$, so $-a \leq 0 \leq x$. ✓
If $x < 0$, then $x = -|x| = (-1) \cdot |x| \geq (-1) \cdot a = -a$. Also, $a \geq 0 \Rightarrow x < 0 \leq a$. ✓
Conversely, suppose $-a \leq x \leq a$. If $x \geq 0$, then $|x| = x \leq a$. ✓
If $x < 0$, then $|x| = -x$, and $-x = (-1) \cdot x \leq (-1) \cdot (-a) = a$. ✓

(iii) Exercise.

(iv) We have, by (ii), $-|x| \leq x \leq |x| \wedge -|y| \leq y \leq |y|$, hence
$-(|x| + |y|) = -|x| + (-|y|) \leq x + y \leq |x| + |y|$, hence $|x+y| \leq |x| + |y|$, by (ii) again. ■

Def. (Interval) Let $(X, \leq)$ be a nonempty set with a linear order relation $\leq$.
A subset $I \subset X$ is called an <u>interval</u> (in X), when

$$\forall x, y, z \in X,\ (x \in I \wedge y \in I \wedge x < z \wedge z < y) \Longrightarrow z \in I.$$

2. COMPLETENESS AXIOM

Def. Let $(X, \leq)$ be a nonempty set with linear ordering $\leq$, let $S \subset X$.
1) Element $a \in X$ is called a lower bound for S, when $a \leq s$, $\forall s \in S$.
2) If S has a lower bound, we say S is bounded below.
3) Element $a \in X$ is called an upper bound for S, when $s \leq a$, $\forall s \in S$.
4) If S has an upper bound, we say S is bounded above.
5) Element $a \in X$ is called the minimal element of S (or minimum of S), when
$$a \in S \wedge (\forall s \in S, a \leq s).$$
6) Element $a \in X$ is called the maximal element of S (or maximum of S), when $a \in S \wedge (\forall s \in S, s \leq a)$.

Examples: Closed vs open intervals, $\mathbb{N}$, $\{\frac{1}{n} : n \in \mathbb{N}_+\}$.

Def. Let $(X, \leq)$ be a nonempty set w/ linear order $\leq$, let $S \subset X$, $S \neq \phi$ be bounded.
1) Element $\alpha \in X$ is called the infimum (or greatest lower bound) of S, when
$(\forall s \in S, \alpha \leq s) \wedge [\forall \beta \in X, \alpha < \beta \Rightarrow (\exists s \in S \text{ s.t. } s < \beta)]$.

2) Element $\alpha \in X$ is called the supremum (or least upper bound) of S, when
$(\forall s \in S, s \leq \alpha) \wedge [\beta \in X, \beta < \alpha \Rightarrow (\exists s \in S \text{ s.t. } \beta < s)]$.

Examples: $\{\frac{1}{n} : n \in \mathbb{N}_+\}$; $[0, \sqrt{2}] \cap \mathbb{Q}$ in $X = \mathbb{Q}$; $[0, \sqrt{2})$ in $\mathbb{R}$; if $\max S$ exists, then $\sup S = \max S$. (!)

Def. We say that a nonempty linearly ordered set $(X, \leq)$ satisfies Completeness Axiom, when every nonempty bounded above subset of X has a least upper bound.

(!) **Def.** The field $\mathbb{R}$ of real numbers is defined as the smallest (w/r/to inclusion) ordered field satisfying the Completeness Axiom.

Archimedean Property of $\mathbb{R}$

Thm. The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Pf. Suppose otherwise, and let $\alpha = \sup \mathbb{N}$. Then, $\alpha - 1$ is not an upper bound for $\mathbb{N}$, so $\exists n_0 \in \mathbb{N}$ st. $\alpha - 1 < n_0$. But then $\alpha < n_0 + 1$. $\lightning$ $\blacksquare$

Thm. TFAE:
(i) $\mathbb{N}$ is not bounded above.
(ii) $\forall x \in \mathbb{R}\ \exists n \in \mathbb{N}$ s.t. $x < n$.
(iii) $\forall x, y \in \mathbb{R},\ x > 0 \Rightarrow \exists n \in \mathbb{N}$ s.t. $n \cdot x > y$
(iv) $\forall x \in \mathbb{R},\ x > 0 \Rightarrow \exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < x$.

Pf. (i) $\Rightarrow$ (ii) ✓
(ii) $\Rightarrow$ (iii): Let $x \in \mathbb{R}_+$, $y \in \mathbb{R}$ be arbitrary. By (ii), $\exists n_0 \in \mathbb{N}$ s.t. $\frac{y}{x} < n_0$.
Then, as $x > 0$, $\frac{y}{x} \cdot x < n_0 \cdot x$. ✓
(iii) $\Rightarrow$ (iv): Let $x \in \mathbb{R}_+$ be arbitrary. By (iii), $\exists n_0 \in \mathbb{N}_+$ s.t. $n_0 \cdot x > 1$.
Then, $n_0 > 0 \Rightarrow \frac{1}{n_0} > 0 \Rightarrow n_0 \cdot x \cdot \frac{1}{n_0} > 1 \cdot \frac{1}{n_0}$; i.e., $x > \frac{1}{n_0}$. ✓
(iv) $\Rightarrow$ (i): Suppose $\alpha \in \mathbb{R}$ is s.t. $\alpha \geq n$, $\forall n \in \mathbb{N}$. Then, $\alpha \geq 1 > 0$, and for all $n \in \mathbb{N}_+$, $\frac{1}{n} \leq \frac{1}{\alpha}$, contradicting (iv). ☒

Thm. For every $s \in \mathbb{R}$, $s > 0 \Rightarrow \exists x \in \mathbb{R}$ s.t. $x^2 = s$.

Pf. Given $s \in \mathbb{R}_+$, let $S := \{x \in \mathbb{R} \mid x \geq 0 \wedge x^2 < s\}$.
Then, $S \neq \emptyset$ as $0 \in S$, and S is bdd above (indeed,
$s > 0 \Rightarrow s+1 > 1 \Rightarrow (s+1)^2 > s+1 \Rightarrow s+1 \notin S$).
Let $\alpha := \sup S$. We claim that $\alpha^2 = s$. Proof by contradiction:
I. Suppose $\alpha^2 < s$.
Then, $s - \alpha^2 > 0$, so $\exists n_1 \in \mathbb{N}$ s.t. $s - \alpha^2 > \frac{1}{n_1}$, or $\alpha^2 + \frac{1}{n_1} < s$.
Now, if we find $n_2 \in \mathbb{N}$ s.t. $(\alpha + \frac{1}{n_2})^2 \leq \alpha^2 + \frac{1}{n_1}$, then $\alpha + \frac{1}{n_2} \in S$, contradicting definition of α.
So, it suffices to find $n_2 \in \mathbb{N}_+$ s.t. $\alpha^2 + \frac{2\alpha}{n_2} + \frac{1}{n_2^2} \leq \alpha^2 + \frac{1}{n_1}$, or $\frac{2\alpha}{n_2} + \frac{1}{n_2^2} \leq \frac{1}{n_1}$.
Since $1 \leq n_2$ then $\frac{2\alpha}{n_2} + \frac{1}{n_2^2} \leq \frac{2\alpha+1}{n_2}$. Choosing $n_2 \geq n_1(2\alpha+1)$ does the job. ✓

II. Suppose then that $s < \alpha^2$.
Then, $\exists n_1 \in \mathbb{N}_+$ s.t. $\alpha^2 - s > \frac{1}{n_1}$, or $\alpha^2 - \frac{1}{n_1} > s$.
Again, we look for $n_2 \in \mathbb{N}_2$ s.t. $(\alpha - \frac{1}{n_2})^2 \geq \alpha^2 - \frac{1}{n_1}$, b/c for such n_2 we get that $\alpha - \frac{1}{n_2} \notin S$ and hence $\sup S \leq \alpha - \frac{1}{n_2}$, contradicting definition of α.
Need $\alpha^2 - \frac{2\alpha}{n_2} + \frac{1}{n_2^2} \geq \alpha^2 - \frac{1}{n_1}$, or $\frac{1}{n_1} \geq \frac{2\alpha}{n_2} - \frac{1}{n_2^2}$.
Now, if $\frac{1}{n_1} \geq \frac{2\alpha}{n_2}$, then also $\frac{1}{n_1} \geq \frac{2\alpha}{n_2} - \frac{1}{n_2^2}$, so suffices to have
$n_2 \geq n_1 \cdot 2\alpha$. ☒

Corollary. For every prime number p, there exists $x_p \in \mathbb{R}$ s.t. $x_p^2 = p$. Hence, $\mathbb{Q} \subsetneq \mathbb{R}$.

Density of $\mathbb{Q}$ in $\mathbb{R}$

Thm. If $x, y \in \mathbb{R}$, $x<y$, then there is $q \in \mathbb{Q}$ s.t. $x<q<y$.

Pf. By Archimedean Principle, $y>x \Rightarrow y-x>0 \Rightarrow \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \frac{y-x}{2}$. Fix such n_0. Then, $\exists k_0 \in \mathbb{N}$ s.t. $x < \frac{k_0}{n_0} < y$. Indeed, $\exists k \in \mathbb{N}$ s.t. $k > n_0 \cdot x$. Let k_0 be the minimal such k (exists, by Well-ordering Principle). Then, $k_0 - 1 \le n_0 \cdot x$, so $\frac{k_0}{n_0} \le x + \frac{1}{n_0} < x + \frac{y-x}{2} = \frac{y+x}{2} < \frac{2y}{2} = y$. ■

Thm. If $x, y \in \mathbb{R}$, $x<y$, then there is $s \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $x<s<y$.

Pf. By above theorem, $\exists q \in \mathbb{Q}$ s.t. $\frac{x}{\sqrt{2}} < q < \frac{y}{\sqrt{2}}$. Then $s := q\sqrt{2}$ is good. ■

Thm. (Nested Interval Principle) Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ be a nested sequence of closed intervals in $\mathbb{R}$. Then, $\bigcap_{k \ge 1} I_k \ne \emptyset$.

Proof: For $k \in \mathbb{N}_+$, let a_k denote the left end-point of I_k, and b_k - the right one. Then, the set $\{a_k \mid k \in \mathbb{N}_+\}$ is bdd above (for instance, by b_1) and the set $\{b_k \mid k \in \mathbb{N}_+\}$ is bdd below (by a_1).
Thus, $\alpha := \sup\{a_k \mid k \in \mathbb{N}_+\}$, $\beta := \inf\{b_k \mid k \in \mathbb{N}_+\}$ are well-defined.
Claim: $\alpha \le \beta$.
For a proof by contradiction, suppose $\beta < \alpha$. Then, β is not an upper bound for $\{a_k \mid k \ge 1\}$, so we can pick a_{k_1} st. $\beta < a_{k_1} \le \alpha$. Then, in turn, a_{k_1} is not a lower bound for $\{b_k \mid k \ge 1\}$, so we can pick b_{k_2} st. $b_{k_2} < a_{k_1}$. Let $k_0 := \max\{k_1, k_2\}$. Then, $b_{k_0} \le b_{k_2} < a_{k_1} \le a_{k_0}$ (by nestedness of the interval sequence), which contradicts $a_{k_0} \le b_{k_0}$. ↯
Now, by construction, $\frac{\beta+\alpha}{2}$ is greater than or equal to a_k, $\forall k \ge 1$, and less than or equal to b_k, $\forall k \ge 1$, hence $\frac{\alpha+\beta}{2} \in I_k$, $\forall k \ge 1$. ■

IV. Infinite Sequences & Series (cf. Ch. 4-7)

1. Limits of Sequences

Def. A sequence $(a_n)_{n=1}^{\infty}$ is said to converge to $L\in\mathbb{R}$, when
$$\forall \varepsilon>0\ \exists N\in\mathbb{N} \text{ s.t. } \forall n\geq N,\ |a_n-L|<\varepsilon.$$
If $(a_n)_n$ converges to L, then L is called its limit, and we write $\lim_{n\to\infty} a_n = L$.
If no such L exists, $(a_n)_n$ is called divergent.

Def. We say that $(a_n)_n$ diverges to ∞, when
$$\forall M\in\mathbb{R}\ \exists N\in\mathbb{N} \text{ s.t. } \forall n\geq N,\ a_n>M. \qquad (\text{Write } \lim_{n\to\infty} a_n=\infty.)$$
We say (a_n) diverges to $-\infty$, when
$$\forall m\in\mathbb{R}\ \exists N\in\mathbb{N} \text{ s.t. } \forall n\geq N,\ m<a_n. \qquad (\lim_{n\to\infty} a_n=-\infty)$$

Thm. 1) Every convergent sequence is bounded (i.e., $\exists M>0$ s.t. $\forall n$, $|a_n|\leq M$).
2) Every sequence divergent to ∞ is bounded below.
3) Every sequence divergent to $-\infty$ is bounded above.

Pf. 1) Suppose $\lim_{n\to\infty} a_n = L$. Let $N_0\in\mathbb{N}$ be s.t. $\forall n\geq N_0$, $|a_n-L|<1$.
Define then $M := \max\{|a_1|,\dots,|a_{N_0-1}|, |L|+1\}$.
We have $|a_n|\leq M$ for all $n\geq 1$, b/c $||a_n|-|L||\leq |a_n-L|$, by triangle inequality.
2), 3) = Exercise. ∎

Thm. If $\lim_{n\to\infty} a_n$ exists then it is unique.

Pf. Suppose $L_1, L_2\in\mathbb{R}$, $L_1\neq L_2$ both are limits of $(a_n)_n$.
Set $\varepsilon := |L_1-L_2|/2$. Then there exist $N_1, N_2\in\mathbb{N}$ s.t.
$$\begin{cases}\forall n\geq N_1,\ |a_n-L_1|<\varepsilon\\ \forall n\geq N_2,\ |a_n-L_2|<\varepsilon.\end{cases}$$
Let $N_0 := \max\{N_1,N_2\}$. Then $N_0\geq N_1 \wedge N_0\geq N_2$,
hence $|a_{N_0}-L_1|<\varepsilon \wedge |a_{N_0}-L_2|<\varepsilon$. Then,
$|L_1-L_2|\leq |L_1-a_{N_0}|+|a_{N_0}-L_2| < 2\varepsilon < |L_1-L_2|$. ↯ ∎

Thm. (Algebraic Limit Theorem) Let $(a_n)_n, (b_n)_n$ be sequences w/ $\lim_\infty a_n = A$, $\lim_\infty b_n = B$.
Then, (i) $(a_n+b_n)_n$ converges and $\lim_{n\to\infty} a_n+b_n = A+B$.
(ii) $(a_n-b_n)_n$ converges and $\lim_{n\to\infty} a_n-b_n = A-B$.
(iii) $(c\cdot a_n)_n$ converges and $\lim_{n\to\infty} ca_n = c\cdot A$, for any $c\in\mathbb{R}$.
(iv) $(a_n\cdot b_n)_n$ converges and $\lim_{n\to\infty}(a_n\cdot b_n) = A\cdot B$.
(v) If $B\neq 0$, then $\left(\frac{a_n}{b_n}\right)_n$ converges and $\lim_{n\to\infty}(a_n/b_n) = A/B$.

Pf. (i) Let $\varepsilon>0$ be arbitrary. Choose $N_1, N_2\in\mathbb{N}$ s.t. $|a_n-A|<\varepsilon/2$ for all $n\geq N_1$ and $|b_n-B|<\varepsilon/2$ for all $n\geq N_2$. Set $N_0=\max\{N_1,N_2\}$.
Then, $\forall n\geq N_0$, $|(a_n+b_n)-(A+B)|\leq |a_n-A|+|b_n-B|<\varepsilon/2+\varepsilon/2=\varepsilon$. ✓

(iv) Write $|a_nb_n-AB| = |a_nb_n - a_nB + a_nB - AB| \leq |a_n|\cdot|b_n-B| + |a_n-A|\,|B|$.
Let $M>0$ be s.t. $|a_n|\leq M$ for all n.
Now, given $\varepsilon>0$, choose $N_1, N_2\in\mathbb{N}$ s.t. $|a_n-A|<\frac{\varepsilon}{M+|B|}$ for all $n\geq N_1$ and $|b_n-B|<\frac{\varepsilon}{M+|B|}$ for all $n\geq N_2$.
Set $N_0:=\max\{N_1,N_2\}$.
Then, $\forall n\geq N_0$, $|a_nb_n-AB| < M\cdot\frac{\varepsilon}{M+|B|} + \frac{\varepsilon}{M+|B|}\,|B| \leq \varepsilon\cdot\frac{M+|B|}{M+|B|}=\varepsilon$. ✓

(v) Write $\left|\frac{a_n}{b_n}-\frac{A}{B}\right| = \left|\frac{a_nB-b_nA}{b_nB}\right| = \left|\frac{a_nB-AB+AB-b_nA}{b_nB}\right| \leq$

$$\leq \frac{|a_n-A|\cdot|B|}{|b_n|\cdot|B|} + \frac{|A|\cdot|b_n-B|}{|b_n|\cdot|B|}.$$

Since $B\neq 0$, there exists $N_1\in\mathbb{N}$ s.t. $|b_n-B|<\frac{|B|}{2}$ for all $n\geq N_1$, and hence $|b_n|>\frac{|B|}{2}$ for all $n\geq N_1$.
Let $\varepsilon>0$ be arbitrary.
Choose $N_2\in\mathbb{N}$ s.t. $|a_n-A|<\frac{\varepsilon\cdot|B|}{4}$ for all $n\geq N_2$.
Also, choose $N_3\in\mathbb{N}$ s.t. $|b_n-B|<\frac{\varepsilon\cdot|B|^2}{4|A|}$ for all $n\geq N_3$.
Set $N_0:=\max\{N_1,N_2,N_3\}$.
Now, $\forall n\geq N_0$, $|a_n/b_n - A/B| \leq \frac{|a_n-A|}{|b_n|} + \frac{|A|\cdot|b_n-B|}{|b_n|\cdot|B|} < \frac{\varepsilon\cdot|B|}{4}\cdot\frac{2}{|B|} + \frac{|A|\cdot\frac{\varepsilon|B|^2}{4|A|}}{\frac{|B|^2}{2}} =$

$$=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \blacksquare$$

Thm. Let $(a_n)_n, (b_n)_n$ be sequences, $A\in\mathbb{R}$, $r>0$, and $\lim_{n\to\infty} b_n = 0$.
If $|a_n-A|\leq r\cdot|b_n|$ for all but fin. many $n\in\mathbb{N}$, then $\lim_{n\to\infty} a_n = A$.

Pf. Given $\varepsilon>0$, let $N_0\in\mathbb{N}$ be s.t. $|b_n|=|b_n-0|<\varepsilon/r$ $\forall n\geq N_0$. Then, $\forall n\geq N_0$,
$|a_n-A|<r\cdot\varepsilon/r=\varepsilon$. $\blacksquare$

Thm. Let $(a_n)_n, (b_n)_n$ be sequences, with $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$.
(i) If $A<B$, then $a_n < b_n$ for all but fin. many $n\in\mathbb{N}$.
(ii) If $a_n \geq b_n$ for all but fin. many $n\in\mathbb{N}$, then $A\geq B$.

Pf. (i) Suppose $A<B$, and let $\varepsilon = \frac{B-A}{2}$. Choose $N_1, N_2 \in\mathbb{N}$ s.t. $|a_n - A|<\varepsilon$ for all $n\geq N_1$ and $|b_n - B|<\varepsilon$ for all $n\geq N_2$. Set $N_0=\max\{N_1,N_2\}$. Then, $\forall n\geq N_0$, $a_n < A+\varepsilon = A+\frac{B-A}{2} = B+(A-B)+\frac{B-A}{2} = B-\frac{B-A}{2} = B-\varepsilon < b_n$. ✓
(ii) By contradiction – Exercise (!) ∎

Example: No "$A\leq B \Rightarrow a_n\leq b_n$". Set $a_n = \frac{(-1)^n}{n}$, $b_n=0$.

Thm. (Squeeze Theorem) Given sequences $(a_n),(b_n),(c_n)$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, and for all but finitely many n, $a_n\leq b_n\leq c$, it follows that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

Pf. Suppose first that $\lim_\infty a_n = \lim_\infty c_n = L\in\mathbb{R}$, and let $N_1\in\mathbb{N}$ be s.t. $a_n\leq b_n\leq c_n$, $\forall n\geq N$.
Let $\varepsilon>0$ be arbitrary, and pick $N_2, N_3\in\mathbb{N}$ s.t. $L-\varepsilon<a_n<L+\varepsilon$, $\forall n\geq N_2$, and $L-\varepsilon < c_n < L+\varepsilon$, $\forall n\geq N_3$. Set $N_0=\max\{N_1,N_2,N_3\}$. Then, $\forall n\geq N_0$, $L-\varepsilon<a_n\leq b_n\leq c_n<L+\varepsilon$, hence $|b_n - L|<\varepsilon$. ✓
Next, suppose $\lim_{n\to\infty} a_n = \infty$. Let $N_1\in\mathbb{N}$ be as above.
Let $M>0$ be arbitrary, and pick $N_2\in\mathbb{N}$ s.t. $M<a_n$, $\forall n\geq N_2$.
Set $N_0 = \max\{N_1, N_2\}$. Then, $\forall n\geq N_0$, $M<a_n\leq b_n$. Thus $\lim_{n\to\infty} b_n=\infty$. ✓
The case $\lim_\infty a_n = -\infty$ is an exercise. ∎

Thm. Let $q\in\mathbb{R}$. Then, $\lim_{n\to\infty} q^n = \begin{cases} \infty, & \text{if } q>1 \\ 1, & \text{if } q=1 \\ 0, & \text{if } |q|<1. \end{cases}$

Lemma (Bernoulli's Inequality) If $a\geq -1$, then $(1+a)^n \geq 1+na$, $\forall n\in\mathbb{N}$.

Pf. (Induction on n): $n=0$: $(1+a)^0 = 1\geq 1+0$. ✓
For $k\geq 0$, $(1+a)^{k+1} = (1+a)^k\cdot(1+a) \geq (1+ka)\cdot(1+a) = 1+a+ka+ka^2 \geq 1+(k+1)a$.
ind. hypothesis
∎

Pf. of Thm.: If $q>1$, then $q^n = (1+(q-1))^n \geq 1+n(q-1)$ and $\lim_{n\to\infty} cn = \infty$ for any $c>0$, hence the claim follows from Squeeze Thm. ✓

If $q=0$ or $q=1$, then (q^n) is a constant sequence.

Suppose then that $q \in (0,1)$. Then, $1/q > 1$, by axioms of ordered field, and hence $\frac{1}{q^n} = \left(\frac{1}{q}\right)^n \geq 1+n\cdot\left(\frac{1}{q}-1\right) = 1+n\cdot\left(\frac{1-q}{q}\right)$.

Let $\varepsilon>0$ be arbitrary, and choose $N \in \mathbb{N}$ s.t. $1+N\cdot\left(\frac{1-q}{q}\right) > \frac{1}{\varepsilon}$ (by Arch. Principle). Then, $\forall n \geq N$, $\frac{1}{q^n} \geq 1+n\cdot\left(\frac{1-q}{q}\right) \geq 1+N\cdot\left(\frac{1-q}{q}\right) > \frac{1}{\varepsilon}$, hence $0<q^n<\varepsilon$, and so $|q^n - 0| < \varepsilon$.

Finally, suppose $q \in (-1,0)$. Then, by above, $\lim_{n\to\infty} |q^n| = \lim_{n\to\infty} |q|^n = 0$, and hence $\lim_{n\to\infty} q^n = 0$, by the lemma below. ∎

Lemma. For any sequence (a_n), $\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$.

Pf. Suppose $\lim_{n\to\infty} a_n = 0$. Let $\varepsilon>0$ be arbitrary, and choose $N\in\mathbb{N}$ s.t. $\forall n\geq N$, $|a_n - 0| < \varepsilon$. Then, $\forall n \geq N$, $||a_n| - 0| = ||a_n|| = |a_n| = |a_n - 0| < \varepsilon$.

Conversely, suppose $\lim_{n\to\infty} |a_n| = 0$. Let $\varepsilon>0$ be arbitrary, and choose $N\in\mathbb{N}$ s.t. $\forall n \geq N$, $||a_n| - 0| < \varepsilon$. Then, $\forall n\geq N$, $|a_n - 0| = |a_n| = ||a_n|| = ||a_n| - 0| < \varepsilon$. ∎

Thm. (i) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(ii) $\forall c>0$, $\lim_{n\to\infty} \sqrt[n]{c} = 1$.

Pf. (i) Let $\varepsilon \in (0,1)$ be arbitrary. Let $N\in\mathbb{N}$ be s.t. $N > \frac{16}{\varepsilon^2}$. Then, $\forall n \geq N$, $n > \frac{16}{\varepsilon^2}$, and if n is even, then $(1+\varepsilon)^{n/2} > n\cdot\frac{\varepsilon}{2}$ (by Bernoulli), hence $(1+\varepsilon)^n > n^2\left(\frac{\varepsilon}{2}\right)^2 > n$ (by above); if n is odd, then $(1+\varepsilon)^{\frac{n-1}{2}} > (n-1)\cdot\frac{\varepsilon}{2}$ (— " —), hence $(1+\varepsilon)^n \underset{1+\varepsilon>1}{>} (1+\varepsilon)^{n-1} > \left(\frac{n-1}{2}\cdot\varepsilon\right)^2 > \left(\frac{n}{4}\cdot\varepsilon\right)^2 > n$.

In any case, $\sqrt[n]{n} < 1+\varepsilon$.

On the other hand, $n>1 \Rightarrow 1-\varepsilon < \sqrt[n]{n}$, and thus $|\sqrt[n]{n} - 1| < \varepsilon$, $\forall n \geq N$. ✓

(ii) Given $c>0$, we have $c \leq n$ for all but fin. many n, and hence $\sqrt[n]{c} \leq \sqrt[n]{n}$ for all but fin. many n. The claim thus follows by Squeeze Thm. ∎

Thm. Given a sequence $(a_n)_n$ with non-zero terms, suppose $\left|\frac{a_{n+1}}{a_n}\right| \leq q$ for some $q<1$ for all but finitely many n. Then, $\lim_{n\to\infty} a_n = 0$.

Pf. Let $N_0 \in \mathbb{N}$ be s.t. $\left|\frac{a_{n+1}}{a_n}\right| \le q$, $\forall n \ge N_0$, where $q \in [0,1)$ is a constant.
Then, $|a_{N_0+1}| \le |a_{N_0}| \cdot q$,
$|a_{N_0+2}| \le |a_{N_0+1}| \cdot q \le |a_{N_0}| \cdot q^2$,
$\ldots$ $|a_{N_0+k}| \le |a_{N_0}| \cdot q^k$, for all $k \in \mathbb{N}_+$.
Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} q^n = 0$, we can choose $N_1 \ge N_0$ s.t. $q^{n-N_0} < \frac{\varepsilon}{|a_{N_0}|}$ for all $n \ge N_1$. Then, $\forall n \ge N_1$,
$$|a_n| = |a_{N_0+(n-N_0)}| \le |a_{N_0}| \cdot q^{n-N_0} < |a_{N_0}| \cdot \frac{\varepsilon}{|a_{N_0}|} = \varepsilon,$$
which proves that $a_n \xrightarrow[n\to\infty]{} 0$. ∎

Corollary. Given $(a_n)_n$, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = c$ and $|c| < 1$, then $\lim_{n\to\infty} a_n = 0$.

Pf. Suppose $\frac{a_{n+1}}{a_n} \xrightarrow[n\to\infty]{} c$, where $|c| < 1$, and $q \in \mathbb{R}$ be s.t. $|c| < q < 1$. Set $\varepsilon = q - |c|$, and choose $N_0 \in \mathbb{N}$ s.t. $\forall n \ge N_0$, $\left|\frac{a_{n+1}}{a_n} - c\right| < \varepsilon$. Then, $\forall n \ge N_0$, $\left|\left|\frac{a_{n+1}}{a_n}\right| - |c|\right| \le \left|\frac{a_{n+1}}{a_n} - c\right| < \varepsilon$, hence $\left|\frac{a_{n+1}}{a_n}\right| < \varepsilon + |c| = q$, and the result follows from the above thm. ∎

Def. (Subsequence) An (infinite) subsequence of a sequence $(a_n)_{n=1}^{\infty}$ is a composition of functions $(a_n) \circ \varphi$, where $\varphi: \mathbb{N}_+ \to \mathbb{N}_+$ is strictly increasing.
Notation: $(a_{n_k})_{k=1}^{\infty}$.

Thm. Let $(a_n)_{n=1}^{\infty}$ be a sequence.
(i) If $(a_n)_n$ is convergent, then so is every its subseq. $(a_{n_k})_k$, and $\lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n$.

(ii) If $(a_n)_n$ is unbounded above, then there exists a subsequence $(a_{n_k})_k$ of $(a_n)_n$ with $\lim_{k\to\infty} a_{n_k} = \infty$.
(iii) If $(a_n)_n$ is unbounded below, then $\exists (a_{n_k})$ s.t. $\lim_{k\to\infty} a_{n_k} = -\infty$.

Pf. (i) Suppose $\lim_{n\to\infty} a_n = L$, and let $(a_{n_k})_k$ be an arbitrary subseq. of $(a_n)_n$. Given arbitrary $\varepsilon > 0$, choose $N_0 \in \mathbb{N}$ s.t. $\forall n \ge N_0$, $|a_n - L| < \varepsilon$, and choose $K_0 \in \mathbb{N}$ s.t. $n_{K_0} \ge N_0$. Then, $\forall k \ge K_0$, $n_k \ge n_{K_0} \ge N_0$, and so $|a_{n_k} - L| < \varepsilon$. ✓

(ii) Suppose $(a_n)_n$ is unbounded above.
Choose $n_1 \in \mathbb{N}$ s.t. $a_{n_1} > 1$, and for any $k \geq 1$, choose $n_{k+1} > \max\{n_1, ..., n_k\}$ s.t. $a_{n_{k+1}} > \max\{|a_{n_k}|, k+1\}$. (Exists, for else $|a_n| \leq \max\{|a_{n_k}|, k+1\}$, $\forall n \geq 1$.)
Claim: $(a_{n_k})_{k=1}^{\infty}$ diverges to ∞.
Indeed, for any $M > 0$ can choose $K_0 \geq 1$ s.t. $M < K_0 + 1$, and hence $\forall k \geq K_0$, $a_{n_k} > k+1 \geq K_0 + 1 > M$. ✓
(iii) Exercise. ∎

Corollary. Suppose $(a_n)_n$ has two convergent subsequences $(a_{n_k})_k$, $(a_{n_\ell})_\ell$ s.t. $\lim_{k\to\infty} a_{n_k} \neq \lim_{\ell \to \infty} a_{n_\ell}$. Then, $(a_n)_n$ is divergent.

Monotone Sequences

Def. A sequence $(a_n)_n$ is said to be:
- increasing, when $a_n \leq a_{n+1}$, $\forall n$
- decreasing, when $a_{n+1} \leq a_n$, $\forall n$
- monotone, when it is increasing or decreasing.

Thm. (Monotone Convergence) If a sequence $(a_n)_n$ is increasing and bdded above, or decreasing and bdded below, then $\lim_{n\to\infty} a_n$ exists.

Pf. W.l.o.g., assume $a_n \leq a_{n+1}$, $\forall n$, and $a_n \leq M$, for some $M > 0$.
Then $\alpha := \sup\{a_n \mid n \in \mathbb{N}_+\}$ exists. Claim: $\lim_{n\to\infty} a_n = \alpha$.
Let $\varepsilon > 0$ be arbitrary. Then, $\alpha - \varepsilon$ is not an upper bound for $\{a_n \mid n \geq 1\}$, and hence can choose $N_0 \in \mathbb{N}$ s.t. $a_{N_0} > \alpha - \varepsilon$. Since $a_{n+1} \geq a_n$, $\forall n$, then $\forall n \geq N_0$, $\alpha - \varepsilon < a_{N_0} \leq a_n$. Also, by def'n of α, $\alpha + \frac{\varepsilon}{2}$ is an upper bound for $\{a_n \mid n \geq 1\}$, and so $a_n \leq \alpha + \frac{\varepsilon}{2} < \alpha + \varepsilon$, $\forall n$. ∎

Thm. (Bolzano-Weierstrass) Every bounded sequence contains a convergent subsequence.

Pf. Given a bounded sequence $(s_n)_n$, let $a_1, b_1 \in \mathbb{R}$ be s.t. $s_n \in [a_1, b_1]$, $\forall n$. If the interval $[a_1, \frac{a_1+b_1}{2}]$ contains inf. many terms of $(s_n)_n$, then set $a_2 = a_1$, $b_2 = \frac{a_1+b_1}{2}$. Otherwise, set $a_2 = \frac{a_1+b_1}{2}$, $b_2 = b_1$. Recursively, having constructed intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_k = [a_k, b_k]$, if $[a_k, \frac{a_k+b_k}{2}]$ contains inf. many terms of $(s_n)_n$, set $a_{k+1} = a_k$, $b_{k+1} = \frac{a_k+b_k}{2}$; otherwise, set $a_{k+1} = \frac{a_k+b_k}{2}$, $b_{k+1} = b_k$. Write $I_{k+1} = [a_{k+1}, b_{k+1}]$.

Then, $(I_k)_{k=1}^{\infty}$ is a nested sequence of closed intervals. By the Nested Interval Property, $\exists p \in \mathbb{R}$ s.t. $p \in \bigcap_{k=1}^{\infty} I_k$.
Claim: $\exists$ subsequence $(s_{n_k})_{k=1}^{\infty}$ of $(s_n)_n$ s.t. $\lim_{k\to\infty} s_{n_k} = p$.
Indeed, set $s_{n_1} = s_1$, and for any $k \geq 1$,
choose $n_{k+1} > n_k$ s.t. $s_{n_{k+1}} \in I_{k+1}$ (exists by infiniteness of $\{s_n \mid n \geq 1\} \cap I_{k+1}$).
Let $\varepsilon > 0$ be arbitrary.
Choose $K_0 \in \mathbb{N}$ s.t. $\frac{b_1 - a_1}{2^{K_0}} < \varepsilon$. Then, for any $k \geq K_0$,

$$|s_{n_k} - p| \leq \operatorname{diam}(I_k) = \frac{b_1 - a_1}{2^k} \leq \frac{b_1 - a_1}{2^{K_0}} < \varepsilon. \blacksquare$$

<u>Def.</u> A sequence $(a_n)_n$ is called <u>Cauchy</u>, when
$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m, n \geq N$, $|a_m - a_n| < \varepsilon$.

<u>Thm.</u> Let $(a_n)_n$ be a sequence. Then, $(a_n)_n$ is Cauchy iff $\lim_{n\to\infty} a_n \in \mathbb{R}$.

Pf. Suppose $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. Let $\varepsilon > 0$ be arbitrary, and choose $N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon/2$, $\forall n \geq N$. Then, $\forall m, n \geq N$,
$|a_m - a_n| \leq |a_m - L| + |a_n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. ✓

Suppose next that $(a_n)_n$ is a Cauchy sequence.
Claim 1: $(a_n)_n$ is bounded.
Indeed, if $N_0 \in \mathbb{N}$ is s.t. $\forall m, n \geq N_0$, $|a_m - a_n| < 1$, then, for all $n \geq N_0$, $|a_n - a_{N_0}| < 1$, and hence $||a_n| - |a_{N_0}|| < 1$, whence $|a_{N_0}| - 1 < |a_n| < |a_{N_0}| + 1$, $\forall n \geq N_0$.
By B.-W., can choose a subsequence $(a_{n_k})_{k=1}^{\infty}$ s.t. $\lim_{k\to\infty} a_{n_k} =: L \in \mathbb{R}$.
Claim 2: $L = \lim_{n\to\infty} a_n$.

Indeed, for an arbitrary $\varepsilon > 0$, choose $K_0 \in \mathbb{N}$ s.t. $\forall k \geq K_0$, $|a_{n_k} - L| < \varepsilon/2$, and choose $N_0 \geq n_{K_0}$ s.t. $\forall m, n \geq N_0$, $|a_m - a_n| < \varepsilon/2$.
Then, $\forall n \geq N_0$, $|a_n - L| \leq |a_n - a_{N_0}| + |a_{N_0} - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\blacksquare$

2. SERIES

Def. A series $\sum_{n=0}^{\infty} a_n$ is a pair of sequences $((a_n)_{n=0}^{\infty}, (s_n)_{n=0}^{\infty})$, where (a_n) is the sequence of terms of the series, and the (s_n) is its sequence of partial sums, satisfying the relation $s_k = \sum_{n=0}^{k} a_n$ for all $k \in \mathbb{N}$.

We say that the series $\sum_n a_n$ is convergent, when $\lim_{n\to\infty} s_n$ exists. In that case, if $L = \lim_{n\to\infty} s_n$, we write $\sum_n a_n = L$ and say that L is the sum of the series. Otherwise, we say that the series diverges.
If $\lim_{n\to\infty} s_n = \infty$, we write $\sum_n a_n = \infty$ and say that the series diverges to ∞.

Thm. (Divergence Test) If $\sum_n a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Pf. $a_n = s_n - s_{n-1} \xrightarrow[n\to\infty]{} L - L = 0$. ∎

Example: The series $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{n}{2n+1}$ is divergent.
Indeed, $\lim_{n\to\infty} n/_{2n+1} = 1/2$ and hence $\lim_{n\to\infty} (-1)^n \cdot \frac{n}{2n+1} \neq 0$. ✓

Warning: The converse of Div. Test doesn't hold!

Ex. Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$ div. to ∞, even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Pf.: Note that the sequence $(s_n)_{n=1}^{\infty}$ of partial sums is increasing, since all the terms of the series are positive. Thus, to show that $\lim_{n\to\infty} s_n = \infty$, it suffices to find a subsequence $(s_{n_k})_{k=1}^{\infty}$ w/ $\lim_{k\to\infty} s_{n_k} = \infty$.
(Exercise!)

Consider the subsequence $(s_{2^k})_{k=0}^{\infty}$.
We have
$$s_{2^0} = s_1 = 1$$
$$s_2 = 1 + \tfrac{1}{2}$$
$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + 2 \cdot \tfrac{1}{2}$$
$$s_{2^3} = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \dots + \tfrac{1}{8}\right) > 1 + 3 \cdot \tfrac{1}{2}$$
$$\dots$$
$$s_{2^k} > 1 + k \cdot \tfrac{1}{2}.$$
Since $\lim_{k\to\infty} \left(1 + \frac{k}{2}\right) = \infty$, then $\lim_{k\to\infty} s_{2^k} = \infty$. ∎

Thm. (Cauchy Criterion for Convergence) A series $\sum_n a_n$ is convergent iff
$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > m \geq N, \ \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon.$$

Pf. By Cauchy's Condition for sequences, $\sum a_n$ converges iff $(s_n)_n$ is Cauchy.
The latter means $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall m, n \geq N, |s_m - s_n| < \varepsilon$. ∎

Def. We say that a series $\sum_n a_n$ is absolutely convergent, when $\sum_n |a_n|$ converges. If $\sum_n a_n$ converges and $\sum_n |a_n|$ diverges, we say that $\sum_n a_n$ is conditionally convergent.

Thm. If $\sum_n |a_n|$ converges, then so does $\sum_n a_n$.

Pf. Given $\sum_n a_n$, suppose that $\sum_n |a_n|$ is convergent.
Let $\varepsilon > 0$ be arbitrary.
By assumption, we can choose $N_0 \in \mathbb{N}$ s.t. $\forall n > m \geq N_0$, $\left|\sum_{k=m+1}^{n} |a_k|\right| < \varepsilon$.
Then, $\forall n > m \geq N_0$, $\left|\sum_{k=m+1}^{n} a_k\right| \leq \sum_{k=m+1}^{n} |a_k| = \left|\sum_{k=m+1}^{n} |a_k|\right| < \varepsilon$, as required. ∎

Thm. (Geometric Series) Let $q \in \mathbb{R}$. If $|q| < 1$, then $\sum_{n=0}^{\infty} q^n$ is absolutely convergent, and $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$. If $|q| \geq 1$, then $\sum_n q^n$ is divergent.

Pf. Suppose first that $|q| \geq 1$. Then, $\forall n \geq 1$, $|q^{n+1}| = |q^n| \cdot |q| \geq |q^n|$, hence the sequence $(|q^n|)_{n=0}^{\infty}$ is bounded below by 1 and increasing. It follows that $\lim_{n\to\infty} |q^n| \neq 0$, and hence $\lim_{n\to\infty} q^n \neq 0$. Thus divergence of $\sum_n q^n$, by Divergence Test.

Suppose next that $|q| < 1$. The following formula can be easily proved by induction on n: For any $x, y \in \mathbb{R}$, $x^{n+1} - y^{n+1} = (x-y)(x^n + x^{n-1}y + \ldots + y^n)$.
Applying the formula with $x = 1$, $y = q$, we get $1 - q^{n+1} = (1-q)(1 + q + q^2 + \ldots + q^n)$ and hence $\sum_{k=0}^{n} q^k = 1 + q + \ldots + q^n = \frac{1-q^{n+1}}{1-q}$, since $1 - q \neq 0$ by assumption.

Thus, the n^{th} partial sum of $\sum_0^{\infty} q^n$, $s_n = \frac{1-q^{n+1}}{1-q}$ tends to $\frac{1-0}{1-q}$ as $n \to \infty$ (by a thm. on p. 31 of the notes)

Hence, $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$.

As for the convergence of $\sum_{n=0}^{\infty} |q^n|$, note that $\sum_{k=0}^{n} |q|^k = \frac{1-|q|^{n+1}}{1-|q|} \xrightarrow[n\to\infty]{} \frac{1}{1-|q|}$. ∎

Thm. (Comparison Test) Given two series $\sum_n a_n$, $\sum_n b_n$ satisfying $0 \leq a_n \leq b_n$ for all but fin. many n.

(i) If $\sum_n b_n$ converges, then so does $\sum_n a_n$.

(ii) If $\sum_n a_n$ diverges, then so does $\sum_n b_n$.

Pf. (i) Let $N_1 \in \mathbb{N}$ be st. $0 \leq a_n \leq b_n$ for all $n \geq N_1$.

Let $\varepsilon>0$ be arbitrary. Choose $N_0 \geq N_1$ s.t. $\forall n>m\geq N_0$, $|\sum_{k=m+1}^{n} b_k|<\varepsilon$.
Then, $\forall n>m\geq N_0$, $|\sum_{k=m+1}^{n} a_k| \leq \sum_{k=m+1}^{n} |a_k| = \sum_{k=m+1}^{n} a_k \leq \sum_{k=m+1}^{n} b_k = |\sum_{k=m+1}^{n} b_k| < \varepsilon$. ✓

(ii) Let $(s_n)_n$ (resp. $(t_n)_n$) denote the sequence of partial sums of $\sum_n a_n$ (resp. $\sum_n b_n$).
Let $N_1 \in \mathbb{N}$ be s.t. $0 \leq a_n \leq b_n$ for all $n \geq N_1$.
It follows that the sequence $(s_n)_{n=N_1}^{\infty}$ is increasing. Since, by assumption $(s_n)_n$ has no limit, then it follows from the Monotone Conv. Thm, that $(s_n)_n$ is unbounded above. Hence, by monotonicity, $\lim_{n\to\infty} s_n = \infty$.
Hence, $\lim_{n\to\infty} t_n = \infty$, as required. ∎

Thm. (Algebraic Conv. Thm.) Given series $\sum_n a_n$, $\sum_n b_n$, and a constant $c \in \mathbb{R}$, suppose $\sum_n a_n$ and $\sum_n b_n$ converge. Then:
(i) $\sum_n (a_n+b_n)$ converges, and $\sum_n (a_n+b_n) = \sum_n a_n + \sum_n b_n$.
(ii) $\sum_n (c\cdot a_n)$ converges, and $\sum_n (c \cdot a_n) = c \cdot \sum_n a_n$.

Pf. Immediate from Alg. Limit Thm. applied to sequences of partial sums. ∎

Example: Determine convergence of $\sum_{n=0}^{\infty} \frac{n-2^n}{\sqrt{6^n+n^2}}$.

by Bernoulli, $(1+1)^n \geq 1+n$

Set $a_n = \frac{n-2^n}{\sqrt{6^n+n^2}}$. Then, $|a_n| \leq \frac{n+2^n}{\sqrt{6^n}} \leq \frac{2\cdot 2^n}{(\sqrt{6})^n} = 2\cdot\left(\frac{2}{\sqrt{6}}\right)^n$.

Now, $\sum_{n=0}^{\infty}\left(\frac{2}{\sqrt{6}}\right)^n$ converges as $|\frac{2}{\sqrt{6}}|<1$, hence $\sum 2\cdot\left(\frac{2}{\sqrt{6}}\right)^n$ conv. by above thm., hence $\sum_0^{\infty} a_n$ converges absolutely by Comp. Test. ∎

Thm. (Alternating Series) Suppose the sequence $(b_n)_{n=0}^{\infty}$ satisfies the following conditions: $b_n \geq 0$ for all but fin. many n, $b_n \geq b_{n+1}$ for all but fin. many n, and $\lim_{n\to\infty} b_n = 0$. Then, the series $\sum_{n=0}^{\infty}(-1)^n b_n$ converges.

Pf. Let $(s_n)_{n=0}^{\infty}$ denote the sequence of partial sums of $\sum(-1)^n b_n$, and let $N_0 \in \mathbb{N}$ be s.t. $0 \leq b_{n+1} \leq b_n$ for all $n \geq N_0$.
Consider the subsequences $(s_{2m})_{m=0}^{\infty}$ and $(s_{2m+1})_{m=0}^{\infty}$ of $(s_n)_n$. We have, $\forall m \geq N_0/2$, $s_{2(m+1)} = s_{2m} + (-1)^{2m+1} b_{2m+1} + (-1)^{2m+2} b_{2m+2} = s_{2m} \underbrace{- b_{2m+1} + b_{2m+2}}_{\leq 0} \leq s_{2m}$,

and $s_{2(m+1)+1} = s_{2m+1} + (-1)^{2m+2} b_{2m+2} + (-1)^{2m+3} b_{2m+3} = s_{2m+1} + \underbrace{b_{2m+2} - b_{2m+3}}_{\geq 0} \geq s_{2m+1}$;

i.e., (s_{2m}) is decreasing and (s_{2m+1}) is increasing (eventually).

Moreover, for any $n > m \geq N_0/2$, $s_{2n+1} = s_{2n} - b_{2n+1} = \ldots = s_{2m} - \underbrace{(b_{2m+1} - b_{2m+2})}_{\geq 0} - \ldots - \underbrace{(b_{2n-1} - b_{2n})}_{\geq 0} - b_{2n+1} \leq s_{2m}$

and $s_{2n} = s_{2n-1} + b_{2n} = s_{2n-3} + (b_{2n-2} - b_{2n-1}) + b_{2n} = \ldots = s_{2m+1} + \underbrace{(b_{2m+2} - b_{2m+3})}_{\geq 0} + \ldots + \underbrace{(b_{2n-2} - b_{2n-1})}_{\geq 0} + b_{2n} \geq s_{2m-1}$,

Which proves that the increasing sequence (s_{2m+1}) is bdd above and the decreasing sequence (s_{2m}) is bounded below. Thus, by M.C.Thm, there exist $\alpha, \beta \in \mathbb{R}$ s.t. $\alpha = \lim_{m\to\infty} s_{2m}$, $\beta = \lim_{m\to\infty} s_{2m+1}$.

Claim: $\alpha = \beta$.

Indeed, by above, we have $\alpha \geq \beta$, so it suffices to disprove $\alpha > \beta$.

Suppose then that $\alpha > \beta$, and let $\varepsilon := \frac{\alpha - \beta}{3}$.

By assumption, we can choose (even) $N_1 \geq N_0$ s.t. $|b_n| < \varepsilon$ for all $n \geq N_1$, $|s_{2m} - \alpha| < \varepsilon$ for all $m \geq N_1/2$ and $|s_{2m+1} - \beta| < \varepsilon$ for all $m \geq N_1/2$.

Then, $|\alpha - \beta| = |\alpha - s_{N_1} + s_{N_1} - s_{N_1+1} + s_{N_1+1} - \beta| \leq |\alpha - s_{N_1}| + \underbrace{|s_{N_1} - s_{N_1+1}|}_{= |b_{N_1+1}|} + |s_{N_1+1} - \beta| <$

$< 3 \cdot \varepsilon = |\alpha - \beta|$. ↯

Thus, $\lim_{m\to\infty} s_{2m} = \alpha = \lim_{m\to\infty} s_{2m+1}$, and so $\lim_{n\to\infty} s_n = \alpha$. (Exercise!) ∎

Example: $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ is conditionally convergent.

Def. Let $\sum_{n=0}^{\infty} a_n$ be a series. A rearrangement of $\sum_n a_n$ is a series $\sum_{n=0}^{\infty} a_{\sigma(n)}$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection.

Thm. (Riemann) Suppose $\sum_{n=0}^{\infty} a_n$ is conditionally convergent. Then, for any real number α, there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ s.t. the rearrangement $\sum_{n=0}^{\infty} a_{\sigma(n)}$ converges to α.

(Sketch of) Pf. Note first that $\sum_{n=0}^{\infty} |a_n| = \infty$, since $\left(\sum_{n=0}^{N} |a_n|\right)_{N=0}^{\infty}$ is increasing and has no limit.

For any n, define $a_n^+ := \max\{0, a_n\}$, $a_n^- := \max\{0, -a_n\}$. Then, $\forall n$, $a_n^+, a_n^- \geq 0$, and $a_n = a_n^+ - a_n^-$, $|a_n| = a_n^+ + a_n^-$. It follows that $\sum_{n=0}^{\infty} a_n^+ = \sum_{n=0}^{\infty} a_n^- = \infty$.

Indeed, for if $\sum a_n^+ < \infty$, then $\sum a_n^- = \sum (a_n^+ - a_n) = \sum a_n^+ - \sum a_n$ converges, by Alg.Conv.T. and hence $\sum |a_n| = \sum a_n^+ + \sum a_n^- < \infty$. ↯

Similarly, if $\sum a_n^- < \infty$, then $\sum a_n^+ = \sum (a_n + a_n^-) = \sum a_n + \sum a_n^-$ converges, and hence so does $\sum |a_n|$. ↯

Define $(b_k)_k = (a_{n_k})_k$ to be the subsequence of $(a_n)_{n=0}^{\infty}$ consisting of those a_n for which $a_n = a_n^+$, and let $(c_\ell)_\ell = (a_{n_\ell})_{\ell=0}^{\infty}$ be the subsequence of the remaining a_n s (i.e., those for which $a_n = a_n^- \wedge a_n < 0$).

Fix $\alpha \in \mathbb{R}$. Set $\sigma(0) = n_0$, so $a_{\sigma(0)} = b_0$. If $s_0 = a_{\sigma(0)} \le \alpha$, then keep adding terms of (b_k) until $s_{N_1} > \alpha$. This will happen after fin. many steps, as $\sum_k b_k = \sum_n a_n^+ = \infty \, (>\alpha)$. Then, start adding terms of (c_ℓ) until $s_{N_2} < \alpha$. Again, will happen eventually, as $\sum_\ell c_\ell = \sum_n a_n^- = -\infty \, (<\alpha)$. Then, switch back to the consecutive terms of (b_k) until $s_{N_3} > \alpha$. And so on. It follows that $\lim_{N\to\infty} s_N = \alpha$, since, $\forall k$,

$$|s_{N_k} - \alpha| \le |a_{\sigma(N_k)}| \text{ and } \lim_{k\to\infty} a_{\sigma(N_k)} = 0. \quad ∎$$

Thm. If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then $\forall$ bijection $\sigma : \mathbb{N} \leftrightarrow \mathbb{N}$ the rearrangement $\sum_{n=0}^{\infty} a_{\sigma(n)}$ is absolutely convergent and $\sum_0^{\infty} a_{\sigma(n)} = \sum_0^{\infty} a_n$.

Pf. Assume $\sum_0^{\infty} a_n$ is absolutely convergent, and let $\sigma : \mathbb{N} \circlearrowleft$ be a bijection.
Let $\varepsilon > 0$ be arbitrary. Choose $N_0 \in \mathbb{N}$ s.t. $\forall n > m \ge N_0$, $|a_{m+1}| + \dots + |a_n| < \varepsilon/2$ and $\forall n \ge N_0$, $\left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{\infty} a_k\right| < \varepsilon/2$.

Set $K_0 := \max\{\sigma^{-1}(0), \dots, \sigma^{-1}(N_0 - 1)\}$. Then, $\forall k > K_0$, $\sigma(k) \notin \{1, \dots, N_0 - 1\}$; i.e., $\forall k > K_0$, $\sigma(k) \ge N_0$.

Then, $\forall k > \ell \ge K_0 + 1$, $\sum_{j=\ell+1}^{k} |a_{\sigma(j)}| \le t_p - t_{q-1} < \varepsilon/2$, where $t_s = \sum_{j=0}^{s} |a_j|$ and $p = \max\{\sigma(j) : \ell+1 \le j \le k\}$, $q = \min\{\sigma(j) : \ell+1 \le j \le k\}$. Thus, $\sum_{n=0}^{\infty} |a_{\sigma(n)}|$ is convergent.

Finally, $\forall K \ge K_0 + 1$, $\left|\sum_{k=0}^{K} a_{\sigma(k)} - \sum_{n=0}^{\infty} a_n\right| \le \left|\sum_{n=0}^{N_0-1} a_n - \sum_{n=0}^{\infty} a_n\right| + \left(|a_{N_0}| + \dots + |a_{\sigma(J)}|\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, where $J \in \mathbb{N}$ is s.t. $\sigma(J) = \max\{\sigma(k) : 0 \le k \le K\}$. ∎

Def. A decimal expansion of $x \in \mathbb{R}$ is a sequence $(a_n)_{n=0}^{\infty}$, where $a_0 \in \mathbb{Z}$ and $a_n \in \{0, 1, \dots, 9\}$ for all $n \in \mathbb{N}_+$, such that $a_0 \le x \le a_0 + 1$, and

$$a_0 + \frac{a_1}{10} + \dots + \frac{a_k}{10^k} \le x \le a_0 + \frac{a_1}{10} + \dots + \frac{a_k + 1}{10}, \text{ for all } k \in \mathbb{N}_+.$$

Prop. 1) Every decimal expansion represents a real number; i.e., $\sum_{n=0}^{\infty} \frac{a_n}{10^n} \in \mathbb{R}$.
2) Every $x \in \mathbb{R}$ admits a decimal expansion.

Pf. 1) By Comparison Test, since $\sum_{n=1}^{\infty} \frac{a_n}{10^n} \le \sum_{n=1}^{\infty} \frac{9}{10^n}$ and $\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} = 1$. ✓

2) Let $x \in \mathbb{R}$, $x > 0$ be arbitrary.

By Archimedean Principle, $\exists n \in \mathbb{N}$ s.t. $x < n$. Let $a_0 + 1$ denote the least such n (exists by Well-ordering of $\mathbb{N}$). Then, $a_0 \leq x \leq a_0 + 1$.

Next, consider $x - a_0 \in [0,1]$. Since $x - a_0 \geq \frac{0}{10}$ and $x - a_0 \leq \frac{9+1}{10}$, there exists a unique $a_1 \in \{0,1,\dots,9\}$ s.t. $a_0 + \frac{a_1}{10} \leq x \leq a_0 + \frac{a_1+1}{10}$.

Inductively, assume that $a_0 + \frac{a_1}{10} + \dots + \frac{a_{k-1}}{10^{k-1}} \leq x \leq a_0 + \frac{a_1}{10} + \dots + \frac{a_{k-1}+1}{10}$, and let $\alpha = 10^{k-1} \cdot \left(x - \sum_{n=0}^{k-1} \frac{a_n}{10^n}\right)$. Since $\alpha \in [0,1]$, then $\alpha \geq \frac{0}{10}$ and $\alpha \leq \frac{9+1}{10}$, and so $\exists!\, a_k \in \{0,1,\dots,9\}$ s.t. $\frac{a_k}{10} \leq \alpha \leq \frac{a_k+1}{10}$. Hence, after dividing by 10^{k-1},

$$a_0 + \frac{a_1}{10} + \dots + \frac{a_k}{10^k} \leq x \leq a_0 + \frac{a_1}{10} + \dots + \frac{a_k+1}{10^k},$$ as required. ∎

Warning: Decimal expansion representations are not unique, in general.

E.g., $1 = 0.999\dots$, since $\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{9}{10} \cdot \sum_{n=0}^{\infty} \frac{1}{10^n} = \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} = 1$, by Geom. Series Thm.