1. Open, Closed, and Compact Sets

Defⁿ: a set $U \subseteq \mathbb{R}$ is open, when

$$\forall a \in U, \quad \exists r > 0 \quad \text{s.t.} \quad (a-r, a+r) \subseteq U$$

Prop.: (i) $\mathbb{R}$ and $\emptyset$ are open.

(ii) for any $\{U_\lambda\}_{\lambda \in I}$ family of open sets, $\bigcup_{\lambda \in I} U_\lambda$ is open.

(iii) if $U_1, \dots U_k$ are open, then $U_1 \cap \dots \cap U_k$ is open.

proof: (i) $\emptyset$ is open b/c it's not true that $\emptyset$ is not open.

(ii) given $\{U_\lambda\}_{\lambda \in I}$ a family of open sets, and $a \in \bigcup_{\lambda \in I} U_\lambda$

then, let $\lambda_a \in I$ be s.t. $a \in U_{\lambda_a}$

↳ by the openness of U_{λ_a}, can choose $r > 0$ s.t.

$$(a-r, a+r) \subseteq U_{\lambda_a} \subseteq \bigcup_{\lambda \in I} U_\lambda \quad \checkmark$$

(iii) given $U_1 \dots U_k$ open sets and $a \in U_1 \cap \dots \cap U_k$

there are $r_1 \dots r_k > 0$ s.t. $(a - r_j, a + r_j) \subseteq U_j$ for $j = 1 \dots k$

↳ set $r := \min\{r_1, \dots r_k\}$

then, $(a-r, a+r) \subset (a-r_j, a+r_j) \subset \mathcal{U}_j \quad , \quad \forall j$

so $(a-r, a+r) \subseteq \mathcal{U}_1 \cap \dots \cap \mathcal{U}_k$

can't do infinite intersections of open sets

$\blacksquare$

Ex) consider $\mathcal{U}_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{Z}_+$

then, $\mathcal{U}_n$ is open $\forall n \geq 1$.

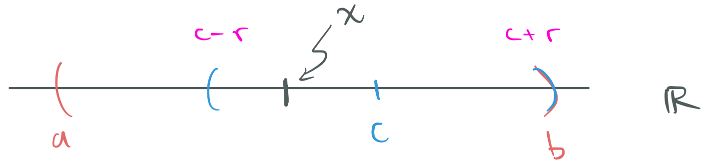
but $\displaystyle\bigcap_{n=1}^{\infty} \mathcal{U}_n = \{0\}$ which is <u>not</u> open.

Prop.: <u>any open interval in $\mathbb{R}$ is an open set.</u>

pf: $I = (-\infty, b)$ or $= (-\infty, \infty)$ or $= (a, \infty)$ or $= (a, b)$

✓ $\because \mathbb{R}$ is open.

- suppose $a, b \in \mathbb{R}$, $a < b$. Consider $I = (a, b)$



in this drawing $r = b-c$

- let $c \in (a,b)$ be arbitrary, then $a < c < b$

  set $r = \min\{c-a, b-c\}$

- <u>claim</u>: $(c-r, c+r) \subseteq (a, b)$

↳ let $x \in (c-r, c+r)$ be arbitrary

then $a \le c - r < x$

$b \ge c + r > x$ by the choice of r

for (a, ∞) choose $r = c - a$

for $(-\infty, b)$ choose $r = b - c$ ■

Def^{n}: a set $F \subseteq \mathbb{R}$ is **closed**, when $\mathbb{R} \setminus F$ is open.

Prop.: (i) $\mathbb{R}$ and $\emptyset$ are closed

(ii) if $\{F_\eta\}_{\eta \in I}$ are closed sets, then $\bigcap_{\eta \in I} F_\eta$ is closed.

(iii) if $F_1 \dots F_k$ are closed, then $F_1 \cup \dots \cup F_k$ is closed.

proof:

↳ ☆ exercise (use the open set properties & deMorg's laws)

Prop.: every closed interval is a closed set.

pf: if $I = [a,b]$ for some $a, b \in \mathbb{R}$, $a \le b$

then $\mathbb{R} \setminus I = (-\infty, a) \cup (b, \infty)$ which is open

$\Longrightarrow$ I is closed. ✓

Ex) $F_n = [0, 1 - \frac{1}{n}]$, for $n \in \mathbb{Z}_+$

can't do inf. unions of closed sets.

$\bigcup_{n=1}^{\infty} F_n = [0,1)$ is not a closed set

b/c $\mathbb{R} \setminus [0,1) = (-\infty, 0) \cup [1, \infty)$ is not open!

$\because$ $1 \in \mathbb{R} \setminus [0,1)$ but $\forall r > 0$, $(1-r, 1+r) \not\subseteq \mathbb{R} \setminus [0,1)$

Thm # 1 a set $X \subseteq \mathbb{R}$ is closed iff.

$\forall x_0 \in \mathbb{R}$, $[\exists (x_n)_{n=1}^{\infty} \subseteq X$ s.t. $\lim_{n\to\infty} x_n = x_0] \Longrightarrow x_0 \in X$

ie, it contains all of its limit points!

Def$^{\underline{n}}$: a pt. $a \in \mathbb{R}$ is a limit point of a set $A \subseteq \mathbb{R}$

when $\exists (x_n)_{n=1}^{\infty} \subseteq A \setminus \{a\}$ s.t. $\lim_{n\to\infty} x_n = a$

proof: $(\Longrightarrow)$

- assume $X \subset \mathbb{R}$ is closed, and let $x_0 \in \mathbb{R}$

  and $(x_n)_{n=1}^{\infty} \subseteq X$ be s.t. $\lim_{n\to\infty} x_n = x_0$

contrad:

- suppose $x_0 \notin X$, then $x_0 \in \mathbb{R} \setminus X$

  choose $\varepsilon > 0$ s.t. $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq \mathbb{R} \setminus X$

  then, $\forall n \in \mathbb{Z}_+$, $|x_n - x_0| \geqslant \varepsilon$

  contradicting $\lim_{n\to\infty} x_n = x_0$

$(\Leftarrow)$

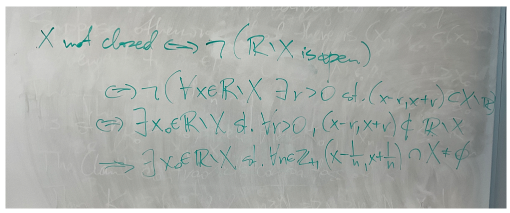
- assume, for every $x_0 \in \mathbb{R}$ s.t. $\exists (x_n)_{n=1}^{\infty} \subseteq X$ w/ $\lim_{n\to\infty} x_n = x_0$

  $x_0 \in X$

contrad:

- suppose X is not closed.

  then $\mathbb{R} \setminus X$ is not open



- choose $x_0 \in \mathbb{R} \setminus X$ s.t.

  $\forall n \in \mathbb{Z}_+$, $\exists x_n \in X$ s.t. $x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$

- then, $\forall n \in \mathbb{Z}_+$, $|x_n - x_0| < \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$

  so $\lim_{n\to\infty} x_n = x_0$

  ↳ but, by assumption, $x_0 \in X$ $\lightning$

  contradicting the choice of $x_0 \in \mathbb{R} \setminus X$ ∎

Def$^{\underline{n}}$: a set $K \subseteq \mathbb{R}$ is compact, when

$\forall (x_n)_{n=1}^{\infty} \subseteq K$, $\exists (x_{n_k})_{k=1}^{\infty} \subseteq (x_n)_{n=1}^{\infty}$ convergent to an element of K.

Prop.: every closed interval is compact.

proof:

- given $I = [a,b]$, w/ $a < b$ and $(x_n)_{n=1}^{\infty} \subseteq [a,b]$
- by Bolzano-Weirstrass, we can choose a subsequence $(x_{n_k})_{k=1}^{\infty} \subseteq (x_n)_n$ s.t. $\lim_{k \to \infty} x_{n_k} = x_0 \in \mathbb{R}$
- by Theorem 1, since $I = \overline{I}$ ie, I is a closed set.

$\Rightarrow x_0 \in I$

∎

Th$^{\underline{m}}$: (Heine-Borel)

A set $K \subseteq \mathbb{R}$ is compact iff. K is closed and bounded

↳ e.g., the ternary Cantor set C is a compact set. (a very weird example)

Proof : $(\Leftarrow)$

- assume K is closed and bounded
- let $(x_n)_{n=1}^{\infty} \subseteq K$ be an arbitrary sequence

By Bolz. – Weir. , we can choose a conv. subseq. $(x_{n_k})_{k=1}^{\infty} \subseteq (x_n)_{n=1}^{\infty}$

let $x_0 = \lim_{k \to \infty} x_{n_k}$

By Thm 1 $\implies x_0 \in K$ ✓

$(\Rightarrow)$

- suppose K is not bounded above

then $\forall n \in \mathbb{Z}_+$, $\exists x_n \in K$ s.t. $x_n \geq n$

↳ claim: (x_n) contains no convergent subseq.

2.

pf: • suppose otherwise that there is

$(x_{n_k})_{k=1}^{\infty} \subseteq (x_n)_{n=1}^{\infty}$ convergent to $x_0 \in \mathbb{R}$

?.

• choose $N_0 \in \mathbb{N}$ s.t. $N_0 \geq x_0 + 1$

$\because n \geq N_0$

$\Rightarrow x_n \geq n \geq N_0 \geq x_0 + 1$

• then $\forall n \geq N_0$, $|x_n - x_0| \geq 1$

in particular, if K_0 is s.t. $n_{K_0} \geq N_0$

then $\forall k \geq K_0$, $|x_{n_k} - x_0| \geq 1$ ↯

exercise: how does this contradict the convergence of $(x_{n_k})_{k=1}^{\infty}$?

- the claim contradicts the compactness of K.

$\star \Rightarrow K$ is bounded.

- let $x_0 \in \mathbb{R}$ and $(x_n)_{n=1}^{\infty} \subseteq K$ be s.t. $\lim_{n\to\infty} x_n = x_0$ Th$^{\underline{m}}$ 1

- by compactness of K, can choose a subseq. $(x_{n_k})_{k=1}^{\infty}$ convergent to $z_0 \in K$

- since $\lim_{n\to\infty} x_n$ exists, then all convergent subseq.s must have the same limit $\Rightarrow x_0 = z_0 \in K$

$\star \Rightarrow K$ is closed

$\blacksquare$

Def$^{\underline{n}}$: a set $U \subseteq \mathbb{R}$ is called an (open) neighbourhood of a point $a \in \mathbb{R}$ when ...: U is open and $a \in U$

ε-δ LIMITS

Def^{n}: given a f^{n} $f: A \to \mathbb{R}$ and a limit point of A $a \in \mathbb{R}$

we say that the limit of f at a exists and equals L

and write $\lim_{x \to a} f(x) = L$, when: ☆

$$\left[\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } \forall x \in A,\ 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon \right]$$

(Ex) show that $\lim_{x \to 1} \frac{1}{x} = 1$)

↳ here $a = 1$, $f(x) = 1/x$, $A = \mathbb{R} \setminus \{0\}$

want to show that ...

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R} \setminus \{0\},\ 0 < |x-1| < \delta \Rightarrow \left| \frac{1}{x} - 1 \right| < \varepsilon$$

↳ before we start ...

this is what we have control over

$$\left| \frac{1}{x} - 1 \right| = \left| \frac{1-x}{x} \right| = \frac{1}{|x|} \cdot |x-1|$$

want to show $\frac{1}{|x|} \leq m$ for all $0 < |x-1| < \delta$

for some m

- assuming that $\delta \leq \frac{1}{2}$, we'll get $\left(1 - \frac{1}{2} < x < 1 + \frac{1}{2}\right)$

then, $\frac{1}{2} < x \Rightarrow \frac{1}{x} < 2 \Rightarrow \frac{1}{|x|} < 2$ b/c $x > 0$

Proof: • let $\varepsilon > 0$ be arbitrary

- choose $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$ $\rightarrow$ like this $\delta \le \frac{1}{2}$ and $\delta \le \frac{\varepsilon}{2}$

then, for all $x \in \mathbb{R}$ s.t. $0 < |x-1| < \delta$

we have $\frac{1}{|x|} < 2$ and so . . .

$$\left|\frac{1}{x} - 1\right| = \ldots = \frac{1}{|x|} \cdot |x-1| < 2 \cdot \delta \le 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

if we had assumed $\delta \le \frac{1}{4}$ we get $\left(1 - \frac{1}{4} < x < 1 + \frac{1}{4}\right)$

$\hookrightarrow$ $x > \frac{3}{4} \Rightarrow \frac{1}{x} < \frac{4}{3} \Rightarrow \frac{1}{|x|} < \frac{4}{3}$ b/c $x > 0$

then the proof would be . . .

- choose $\delta = \min\left\{\frac{1}{4}, \frac{3\varepsilon}{4}\right\}$

then, for all $x \in \mathbb{R}$ s.t. $0 < |x-1| < \delta$

we have $\frac{1}{|x|} < \frac{4}{3}$ and so . . .

$$\left|\frac{1}{x} - 1\right| = \ldots = \frac{1}{|x|} \cdot |x-1| < \frac{4}{3} \cdot \delta \le \frac{4}{3} \cdot \frac{3\varepsilon}{4} = \varepsilon$$

we want to find the ε expression in terms of the δ one

the only thing we can control / play w/

shrinking δ lets us bound the values of x we are working w/.

$\text{Th}^{\underline{m}}$: given $f: A \to \mathbb{R}$ and $a \in \mathbb{R}$ a limit pt. of A

the following conditions are equivalent:

(i) $\lim_{x \to a} f(x) = L$

very useful for proofs!

(ii) $\forall (x_n)_{n=1}^{\infty} \subseteq A \setminus \{a\}$,

if $\lim_{n \to \infty} x_n = a$, then $\lim_{n \to \infty} f(x_n) = L$

proof: (i) $\Rightarrow$ (ii)

- let $(x_n)_{n=1}^{\infty} \subseteq A \setminus \{a\}$ be any seq. s.t. $\lim_{n \to \infty} x_n = a$

contrad

- suppose $\neg \left(\lim_{n \to \infty} f(x_n) = L \right)$

then $\neg \left(\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, |f(x_n) - L| < \varepsilon \right)$

$\Rightarrow$ we can choose $\varepsilon_0 > 0$ where $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t. $|f(x_n) - L| \geq \varepsilon$

- by assumption, $\exists \delta > 0$ s.t. $0 < |x - a| < \delta_0 \Rightarrow |f(x) - L| < \varepsilon_0$

choose $N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0$, $|x_n - a| < \delta_0$

↳ now, $\forall n \geq N_0$, $0 < |x_n - a| < \delta_0$

and hence $|f(x_n) - L| < \varepsilon_0$ ↯.

(ii) $\Rightarrow$ (i)

contrad

- assume (ii) and suppose $\neg \left(\lim_{x \to a} f(x) = L \right)$

implied by this negation ...

[then $\exists\, \varepsilon_0 > 0 \;\; \forall n \in \mathbb{Z}_+ ,\; \exists\, x_n \in A \setminus \{a\}$ s.t.

$|x_n - a| < \frac{1}{n} \;\; \wedge \;\; |f(x_n) - L| \geq \varepsilon_0$]

choice for δ

- the seq. $(x_n)_{n=1}^{\infty} \subseteq A \setminus \{a\}$ converges to a (by squeeze thm w/ $\frac{1}{n}$)

  and so, by (ii) $\lim\limits_{n\to\infty} f(x_n) = L$

  ↳ in particular ...

  $\exists\, N_0 \in \mathbb{N}$ s.t $\forall n \geq N_0 ,\; |f(x_n) - L| < \varepsilon_0$ $\lightning$.

$\blacksquare$

Thm: (Algebraic Limit Theorem)

given $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$, $a \in \mathbb{R}$ is a limit pt. of A
$c \in \mathbb{R}$ constant.

suppose $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} g(x) = L_2$ for some $L_1, L_2 \in \mathbb{R}$.

Then:

(i) $\lim\limits_{x\to a} (f \pm g)(x) = L_1 \pm L_2$

(ii) $\lim\limits_{x\to a} (c\cdot f)(x) = c \cdot L_1$

(iii) $\lim\limits_{x\to a} (f\cdot g)(x) = L_1 \cdot L_2$

also applies for f^ns continuous at a point 'a'

(iv) $\lim_{x \to a} (f/g)(x) = L_1 / L_2$ provided $L_2 \neq 0$

proof: • it suffices to show that for any sequence $(x_n)_{n=1}^{\infty} \subseteq A \setminus \{a\}$

the seq.s $\left(f \pm g(x_n)\right)_{n=1}^{\infty}$, $\left(c \cdot f(x_n)\right)_{n=1}^{\infty}$, $\left(f(x_n) \cdot g(x_n)\right)_{n=1}^{\infty}$, $\left(\frac{f(x_n)}{g(x_n)}\right)_{n=1}^{\infty}$

converge to the expected limits.

↳ known by the proof of A.L.T. for seq.s

Def^{n}: a point $a \in A$ is an isolated point of A

when $\exists \delta > 0$ s.t. $(a - \delta, a + \delta) \cap A = \{a\}$

↳ ie., no other points near it for some open interval

☆ Remark:

↳ for any $a \in A$, it is either an isolated pt. or a limit pt. of A.

we say that a function $f: A \to \mathbb{R}$ is continuous at $a \in A$

when a is an isolated point of A (side case)

OR $\lim_{x \to a} f(x) = f(a)$

↳ trivial ex) any $f: \mathbb{N} \to \mathbb{R}$, $g: \mathbb{Z} \to \mathbb{R}$ are all cts.

ie, every pt. is isolated...

$Th^{\underline{m}}$: given $f: A \to \mathbb{R}$, $a \in A$ the following cond$^{\underline{n}}$s are equiv. :

(i) f is continuous at a

(ii) $\forall \varepsilon > 0$, $\exists \delta > 0$ $\forall x \in A$, $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

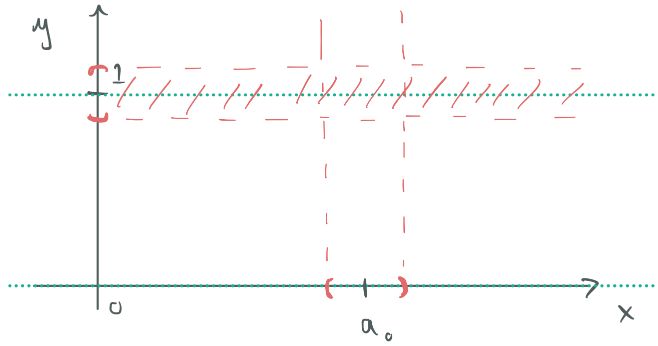
(iii) $\forall (x_n)_{n=1}^{\infty} \subseteq A$, $\lim\limits_{n\to\infty} x_n = a \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(a)$

(iv) $\forall V$ neighborhood of $f(a)$, $\exists U$ neighborhood of a s.t. $f(U) \subseteq V$

↳ Mp' TM #2 type (proof) !

Exercise

Ex) consider $f: \mathbb{R} \to \mathbb{R}$ define $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$



both the rationals & irrationals are <u>dense</u>

<u>claim</u>: $\lim_{x \to a} f(x)$ D.N.E. $\forall a \in \mathbb{R}$

contrad! pf: • if $\lim_{x \to a_0} f(x)$ existed, then

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t. } \forall x \in \mathbb{R} \setminus \{a_0\}, \quad |x - a_0| < \delta \implies |f(x) - L| < \varepsilon$$

• suppose $\lim_{x \to a_0} f(x) = 1$, choose $\varepsilon = 1$

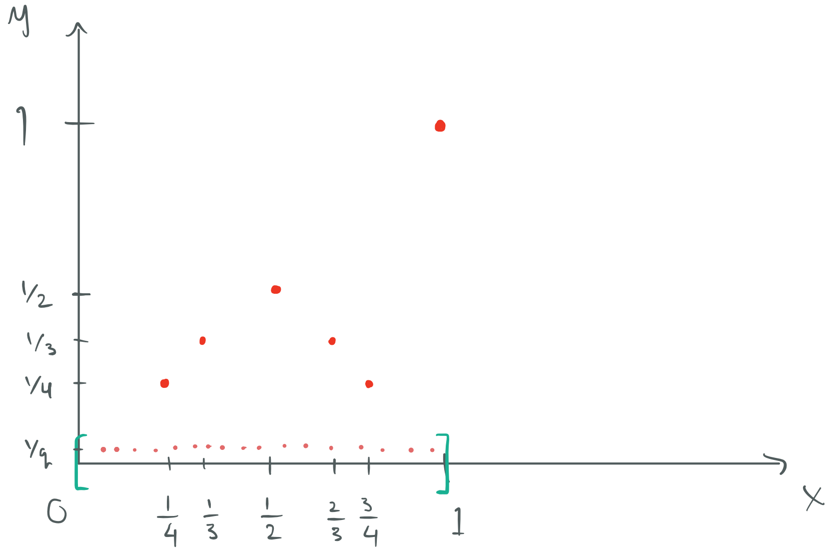
↳ observe that for any $\delta > 0$ there is $x_0 \in \mathbb{Q} \cap (a_0 - \delta, a_0 + \delta) \setminus \{a_0\}$

and hence $\neg \left(\exists \delta > 0, \text{ s.t. } \forall x \, \mathbb{R}, \; 0 < |x - a_0| < \delta \text{ s.t. } |f(x) - 1| < 1 \right)$

because $|f(x_0) - 1| = 1$

$\blacksquare$

Ex) $g: [0,1] \to \mathbb{R}$ defined $g(x) = \begin{cases} 0, & x \in [0,1] \setminus \mathbb{Q} \\ 1/q, & x = \frac{p}{q} \text{ in lowest terms } p, q \in \mathbb{N} \end{cases}$



claim 1: for any $a \in [0,1] \cap \mathbb{Q}$, $\lim_{x \to a} g(x)$ D.N.E.

claim 2: g is continuous at 'a', for any $a \in [0,1] \cap \mathbb{Q}$