Problem Set 3

7

a

Suppose $x_u \geq x_w$. Then $x_u \in (x_w, +\infty)$, so $f(x_w) = w \leq f(x_u) = u$, contradicting that $u < w$. ■

b

$f(x_u) = u < v$ so $x_u \in A$. Similarly $f(x_w) = w > v$ so $x_w \in B$. Also, A and B are bounded by $[x_u, x_w]$. So A, B are non-empty and bounded.

Note that $\forall a \in A, \forall b \in B$, $f(a) < v < f(b)$, so $a < b$. Suppose for the sake of contradiction that $\alpha > \beta$. Then $\exists a \in A$ such that $a > \beta$. But then, a is a lower bound for B greater than β, contradicting the fact that β is a greatest lower bound. So $\alpha \geq \beta$. A symmetric argument gives $\alpha \leq \beta$. So $\alpha = \beta$. ■

c

Suppose for contradiction $f(\alpha) > \inf f(B)$. Then from (a), $\alpha > \inf B = \beta$, contradicting (b). Similarily, $f(\alpha) < \inf f(B)$ is a contradiction. So $f(\alpha = \inf f(B))$. ■

d

We show $v = \inf f(B) =: \gamma$. Suppose $v > \gamma$. Then there exists a $b \in f(B)$ such that $\gamma \leq b < v$. But this is impossible, since $b \in f(B) \implies f(b) > v$. Similarily, $v < \gamma$ is impossible. So $v = \gamma$. ■

8

TODO.

9

Let $A = (a_n)$ be a convergent sequence. Since it is convergent, it is bounded. So $I = \inf a_n$ and $S = \sup a_n$ exists. Suppose for contradiction $S, I \notin A$. Consider $S \neq I$. Construct subsequences K, J that converge to S, I respectively. Then A is a convergent sequence containing subsequences that converge to two different limits; a contradiction. Now consider $S = I$. Since $S, I \notin A$, $I < a_n < S$; a contradiction. ■

10

$a_n = x, b_n = x^2$ gives $\lim a_n - b_n = 0$ and $\lim a_n / b_n = 0$. For the second case, $a_n = x + 1$ and $b_n = x$ gives $\lim a_n / b_n = 1$ but $\lim a_n - b_n = 1$.

11

Recall the reverse triangle inequality, $|a - b| \geq ||a| - |b||$. Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\forall n \geq N, |a_n - L| < \epsilon$. From the reverse triangle inequality, $||a_n| - |L|| \leq |a_n - L| < \epsilon$. ■

12

See https://math.stackexchange.com/a/923055 .

13

Choose $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \epsilon$. Then

$$\begin{aligned} |\frac{a_1 + \cdots + a_n}{n} - a| &= \frac{1}{n}|(a_1 - a) + \cdots + (a_n - a)| \\ &\leq \frac{1}{n}(|a_1 - a| + \cdots + |a_n - a|) \\ &< \frac{1}{n}n\epsilon \\ &= \epsilon \end{aligned}$$

14

Since (a_n) diverges to infinity, there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n > M$ for all $M \in \mathbb{R}$. Consider the finite set $\{a_1, \ldots, a_N\}$. Then $\forall n \geq N$, $a_n \geq a_1, \ldots, a_N$. So $\min\{a_1, \ldots, a_N\} = \min(a_n)$.

15

Consider $\epsilon = \frac{c}{2}$. We have $|a_{n+1} - a_n - c| < \frac{c}{2}$ so $a_{n+1} - a_n > \frac{c}{2}$ Then $a_{n+2} - a_{n+1} + a_{n+1} - a_n > \frac{c}{2} + \frac{c}{2} = c$ and in general, $a_{n+k} - a_n > k \cdot \frac{c}{2}$. So, we can make a_{n+k} arbitrary large.

Let $M > 0$. Choose $N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n > M$, which is possible since we can make a_n arbitrary large (from above). ■

16

Let β be the lower bound of (b_n). Let $M > 0$. Choose $N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n > M - \beta$. Then $a_n + b_n \geq a_n + \beta > M$. ■