Problem Set 4

4

By the Algebraic Limit Theorem, it suffices to show that $\lim_{n\to\infty} \sqrt[n]{|P(n)|} = 1$. We have $\lim_{n\to\infty} |P(n)| = \infty$ so $|P(n)| \geq 1$ for all but finitely many n. We have

$$\begin{aligned} 1 &\leq \sqrt[n]{|P(n)|} \\ &= \sqrt[n]{|a_k n^k + \cdots + a_0|} \\ &\leq \sqrt[n]{n^k(|a_k| + \cdots + |a_0 n^{-k}|)} \\ &= \sqrt[n]{n}^k \sqrt[n]{|a_k| + \cdots + |a_0 n^{-k}|} \end{aligned}$$

Taking the limit of the right side of the inequality as $n \to \infty$, the second factor $\to 0$ and the first $\to 1$. So by Squeeze Theorem, $\sqrt[n]{|P(n)|} = 1$. ■

5

Let $N_0 \in \mathbb{N}$ such that $\forall n \geq N$ we have $|\frac{a_{n+1}}{a_n} - c| < \epsilon$ for all ϵ, namely, $\epsilon < 1$. So $\frac{a_{n+1}}{a_n} > c - \epsilon$, and $\frac{a_{n+2}}{a_{n+1}} \frac{a_{n+1}}{a_n} > (c-\epsilon)^2$. Simlarily, $\frac{a_{n+k}}{a_n} > (c-\epsilon)^k$ and $a_{n+k} > (c-\epsilon)^k a_n$, which $\to \infty$ as $k \to \infty$.

Let $M > 0$. Choose $K \in \mathbb{N}$ such that, $\forall k \geq K$, $a_{N_0+k} > M$. Choose $N = N_0 + K$. Then, for all $n \geq N$, $a_{N+k} > M$. ■

6

TODO.

7

https://math.stackexchange.com/a/1340559

Note that we can show $b_n \geq a_n$ by showing $b_n - a_n \geq 0$.

8

Suppose the sequence (a_n) did converge. Then every one of its subsequences also converges by the theorem proved in class. But (a_n) does not have a convergent subsequence: so (a_n) diverges.

9

We prove the contrapositive. Suppose (a_n) is bounded and it does not converge to b. Then there is an $\epsilon > 0$ such that $\forall N \in \mathbb{N}, \exists n \geq N$ such that $|a_n - b| > \epsilon$.

These a_n's form a subsequence S, which is bounded, so by Bolzano-Weistrass, contains a subsequence S' that converges. But, since $\forall x \in S$, $|s - b| > \epsilon$, S' can't converge to b. $\blacksquare$

10

https://math.stackexchange.com/a/3975498/890112

11

Use Monotone Convergence Theorem.

12

Use Monotone Convergence Theorem.

13

Let $s := \liminf a_n = \limsup a_n$ Suppose for contradiction that $\lim_{n\to\infty} a_n \neq s$. Then, for $\epsilon > 0$, there are infinitely many a_n such that $a_n > \epsilon + s$. Create a subsequence S from such a_n. Since (a_n) is bounded, S is bounded, so by Bolzano-Weistrass, we can create a subsequence S' convergent to some limit L. But, since $\forall x \in S$, $x > \epsilon + s$, so $L > \epsilon + s$. Then, L is a subsequential limit with $L \geq \limsup a_n$; a contradiction. Similarily, there are not infinitely many a_n such that $a_n < \epsilon - s$. So $\lim a_n = s$. $\blacksquare$

14

a

Define $\alpha := \liminf a_n + b_n$. Choose subsequences $a' \subseteq a_n, b' \subseteq b_n$ such that $\lim_\infty a' + b' = \alpha$. Since a', b' are convergent, they are bounded, so by Bolzano-Weistrass, we can choose subsequences $a'' \subseteq a'$ and $b'' \subseteq b'$ that converge to A, B, respectively. By definition of $\liminf$, $A \geq \liminf a_n$ and $B \geq \liminf b_n$. And, since the subsequences of convergent sequences converge to the same limit, $\lim_\infty a'' + b'' = \alpha$. So, $\alpha = \liminf a_n + b_n = a'' + b'' = A + B \geq \liminf a_n + \liminf b_n$. Similarily, we can show $\limsup a_n + b_n \leq \lim supa_n + \limsup b_n$. $\blacksquare$.

b-c

$a_n = \sin(n \cdot \frac{pi/2}{)}$ and $b_n = \sin(n \cdot \frac{\pi}{2} - \pi)$ works: note that $\liminf a_n = -1$ and $\liminf b_n = -1$, but $\liminf a_n + b_n = 0$ (and similarly for $\limsup$).