David Tran
251169871

HW 1

7.3

22. (a) *Proof.* Let $a \in R$ and $A = \{x \in R \mid ax = 0\}$. Let $x, y \in A$.Then $a(x - y) = ax - ay = 0$, so A is a subring of R. Now let $r \in R$. Then $axr = 0r = 0$, so the subring is closed under right-multiplication by R. Thus A is a right ideal of R.

    Let $B = \{x \in R \mid xa = 0\}$. Let $x, y \in B$. Then $(x-y)a = xa-ya = 0$, so B is a subring of R. Now let $r \in R$. Then $rxa = r0 = 0$, so the subring is closed under left-multiplication by R. Thus B is a left ideal of R. □

    (b) *Proof.* Let $L \subseteq R$ a left-ideal of R and define $A = \{x \in R \mid xa = 0, \forall a \in L\}$. Let $x, y \in A$ and $a \in L$. Then $(x-y)a = xa-ya = 0$. So A is a subring of R. Now let $r \in R$. Since L is a left-ideal, $ra \in L$, so $xra = x(ra) = 0$. Thus the subring is closed under right-multiplication by R. Therefore it is a right ideal of R. Also, $rxa = r(xa) = 0$, so it is also closed under left-multiplication. Thus A is also a left ideal, and thus a two-sided ideal of R. □

29. *Proof.* Let $x, y \in R$ such that $x^m = y^n = 0$. Then $(x + y)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k y^{m+n-k}$. Since $x^m = y^n = 0$, the sum is zero for $k \geq m$ or $k \leq n$. Thus $(x+y)^{m+n} = 0$, so $\mathfrak{N}(R)$ is a subring of R. Now let $r \in R$. Then $(rx)^m = r^m x^m = 0$, and similarly $(xr)^m = 0$, so $\mathfrak{N}(R)$ is closed under multiplication by R. Therefore $\mathfrak{N}(R)$ is an ideal of R. □

30. *Proof.* Let $r + \mathfrak{N}(R) \in R/\mathfrak{N}(R)$ be nilpotent. Then $(r + \mathfrak{N}(R))^n = r^n + \mathfrak{N}(R) = 0 + \mathfrak{N}(R)$. for some positive integer n. Then $r \in \mathfrak{N}(R)$, so $r = \bar{0}$. Thus only $\bar{0} + \mathfrak{N}(R)$ is nilpotent in $R/\mathfrak{N}(R)$, so $\mathfrak{N}(R/\mathfrak{N}(R))$ is trivial. □

7.4

13. *Proof.* Suppose $\varphi^{-1}(P) \neq R$ and let $ab \in \varphi^{-1}(P)$. Then $\varphi(ab) = \varphi(a)\varphi(b) \in P$, so $\varphi(a) \in P$ or $\varphi(b) \in P$. Thus $a \in \varphi^{-1}(P)$ or $b \in \varphi^{-1}(P)$, so $\varphi^{-1}(P)$ is a prime ideal of R.

    Thus, if $R \subseteq S$ and φ is the inclusion homomorphism, if P is a prime ideal of S, then either $\varphi^{-1}(P) = R$, in which case $R \subseteq P$ so $R \cap P = R$, or $\varphi^{-1}(P)$ is a prime ideal of R, in which case $R \cap P = \varphi^{-1}(P)$; a prime ideal of R. □

24. *Proof.* Let R be a Boolean ring and $(A) \subseteq R$ an ideal of R finitely generated by $A \subseteq R$. Suppose $x, y \in A$ are distinct. Consider $z = x + y + xy \in (A)$. Then $xz = x + xy + xy = x$, and similarly $yz = y$. Thus z alone generates (A). By induction, any number of distinct elements in A can be generated by a single element. Thus (A) is a principal ideal. □

35. Let $\mathcal{I}$ be the family of ideals in R not containing A and $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ a chain of ideals in $\mathcal{I}$. Then $J = \bigcup_{i=1}^{n} I_i$ is closed under subtraction since for any $a, b \in J$, a, b is contained in some ideals I_a, I_b, respectively. The maximal (by inclusion) of I_a, I_b thus contains both and is itself is an ideal and thus closed under subtraction. J is also closed under multiplication in R since each I_i is closed under multiplication in R. So J is an ideal. $J \in \mathcal{I}$ since if it weren't, then J would contain A. If J contains A, then each element of A is contained in some ideal I_i, and the maximal (by inclusion) of all such ideals would contain all of A, contradicting that each $I_i \in \mathcal{I}$.

    So J each chain of ideals in $\mathcal{I}$ has an upper bound in $\mathcal{I}$. By Zorn's lemma, $\mathcal{I}$ has a maximal element M.